Scalability and Peer Churning in IP-TV: an Analytical Insight

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***Abstract*— Peer-to-peer (P2P) technology for TV broadcasting over the Internet is becoming more and more popular in the very last years.**

This paper introduces a network-wide metric to assess the efficiency of P2P streaming systems and develops a mathematical model to explain:

(i) **the scalability of such architectures with the number of peers, as evidenced by recent measurements;**
(ii) **the initial decrement of efficiency (hence, quality) when a sharp increase in the number of peers in system occurs, as reported by experimental data.**

As for the second point, the proposed model builds upon the fundamental remark that when a peer first joins the system, it has no video content to share with others: its upload contribution is null for an initial time interval and the new peer behaves as a free rider.

Three situations concerning the system reaction to the requests of the new entering peers are examined: full compensation; partial compensation; no reaction at all. Depending on the system answer and on its extent, system efficiency is shown to exhibit different time trends.

***Index Terms*— peer-to-peer video streaming, analytical model, indirect quality monitoring**

I. Introduction

P2P networks have received much attention in recent years. The underlying communication technology is emerging as scalable and powerful, above all in the IP-TV field.

In this work, we focus on modeling and characterizing a P2P streaming system with the intent of providing an analytical explanation to the dependence of its performance on the number of peers in system. Additionally, we set the basis to capture the effect on system performance of a relevant number of new peers joining the overlay in a short time interval. To the best of our knowledge, these represent modeling issues that have not been investigated by previous studies on P2P video streaming systems.

A few modeling works are present in literature in this subject area: an interesting analytical study is presented in [1], where the authors highlight the effect of some factors such as upload/download capacity heterogeneity and playback delay on system behavior. A new strategy, that represents a good trade-off to ensure both playback continuity and low startup delay in P2P streaming systems, is described in [2]. The authors of [2] also present a model that allows to compute the distribution of what each peer has in its buffer. Based on this information, they are able to compare different chunk selection strategies and to understand the relationships between important system metrics. More recently, [3] evaluates the performance of a few peer selection strategies that take advantage of the proximity-aware notion. In [4], a queuing network model is proposed, to analytically investigate the performance of multi-channel P2P streaming systems.

Ultimately however, one of the most important goals of any P2P video streaming architecture is to guarantee a good quality to the peers viewing the video; hence the necessity to trace it. As the study in [5] well evidences, quality monitoring is highly valuable from different perspectives: service providers need it to understand how well their system is behaving, so as to introduce appropriate countermeasures when quality degrades. This eliminates having peers that experience prolonged delays and poor quality in the delivered videos, and quit the system. Monitoring the service of various P2P streaming providers could also be of interest to third parties, to independently assess the fraction of satisfied users in different platforms.

In this work we focus on quality in P2P streaming systems and on its indirect monitoring via a network indicator. Our starting point is represented by some measurements performed on a small mesh-based P2P system that broadcasts test television channels, StreamerOne [6]. In contrast to [5], where a completely experimental approach is taken, we start from real data to identify a system-wide efficiency indicator and then determine its analytical dependency on the number of peers simultaneously viewing the same channel. To this end, we develop a fluid model that allows to demonstrate the scalability of P2P streaming systems with the number of peers concurrently viewing the same channel, as highlighted by recent measurements [7], [8], [9]. The model also allows to understand the extent to which the existing peers in the P2P streaming system can successfully cope with peer churns.

The rest of the paper is organized as follows. Section II introduces the efficiency indicator and formalizes its analytical relationship to the number of peers in system. System efficiency is computed in three different cases where the number of connected peers increases. Section III gives a brief description of the examined P2P streaming architecture; it then presents some experimental and theoretical results. Finally, Section IV summarizes the main findings.

II. Analytical model for efficiency estimate

The intent of any P2P streaming architecture is to guarantee its users a good viewing experience, as dissatisfaction might ultimately lead customers to abandon the overlay. To achieve such goal, the first step is to devise a content distribution mechanism that effectively employs the peers bandwidths. We therefore choose to monitor the variations in upload and download rates that the peers exhibit system-wide, in an effort to indirectly capture how well the system as a whole is behaving.

We consider a general reference model for the P2P system, without requiring any specific assumption on the underlying overlay topology (i.e., mesh, tree, hybrid).

The elementary building block of our analysis is represented by the ratio between the upload and the download rates effectively exploited by the single peer at time t. For the i-th peer we denote such ratio by $q_i(t)$, and by $u_i(t)$ and $d_i(t)$ the corresponding upload and download rates, respectively, so that

$$q_i(t) = \frac{u_i(t)}{d_i(t)}\,, \quad d_i(t) > 0\,. \tag{1}$$

It is immediate to observe that $q_i(t) = 0$ indicates no content is provided to other users, because either the i-th peer owns no content or it cannot provide any, being behind a NAT or firewall; in contrast, $q_i(t) \geq 1$ indicates that the i-th peer favorably relays the content it owns to other peers.

In essence, the $q_i(t)$ ratio is a reasonable indicator for the peer's capability to contribute to the correct functioning of the whole P2P system.

Next, we define the efficiency $Q(t)$ of the system at time t as:

$$Q(t) = \frac{\sum_{i=1}^{N(t)} q_i(t)}{N(t)}, \tag{2}$$

where $N(t)$ is the number of peers in system at time t, $N(t) > 0$. $Q(t)$ is the average of $q_i(t)$ taken over the entire population of active peers: as such, it is also employed in P2P file sharing systems.

Our aim is to capture how variations in the number of peers, $N(t)$, affect the system indicator $Q(t)$: more generally, we want to identify the analytical relation between $N(t)$ and $Q(t)$. To pursue this goal, we resort to model the P2P system in the continuous domain. Accordingly, with a standard time-interpolation, a generic peer of the P2P overlay is represented by a point x in the $[0, N(t)]$ interval. Moreover, we let $u(t,x)$ and $d(t,x)$ represent the upload and download rate of peer x at time t, respectively, and given $d(t,x) > 0$, we replace (1) by the contribution provided by peer x at time t:

$$q(t,x) = \frac{u(t,x)}{d(t,x)}\,, \quad d(t,x) > 0\,. \tag{3}$$

As a consequence, $Q(t)$ previously given by (2) turns into:

$$Q(t) = \frac{\int_0^{N(t)} q(t,x)dx}{N(t)}. \tag{4}$$

Before stepping over, we recall the fundamental law holding in a generic P2P system: given that S indicates the video server upload rate, at an arbitrary time t the following balance equality holds

$$\int_0^{N(t)} u(t,x)dx + S = \int_0^{N(t)} d(t,x)dx\,. \tag{5}$$

Last relation states that the upload rates of the peers and the server streaming rate concur together to determine the overall download rates in system.

We next evaluate the derivative of $Q(t)$: as thoroughly detailed in the Appendix, it is given by:

$$Q'(t) = -\frac{N'(t)}{N(t)}Q(t) + \frac{1}{N(t)}\int_0^{N(t)} \frac{\partial}{\partial t}q(t,x)dx + \\ + \frac{N'(t)}{N(t)}q(t,N(t)). \tag{6}$$

Let us now take into account an increase in the peers number ($N'(t) > 0$) and focus on the newcomers among the peer population at time t. Such new peers could actually be

- good users entering the system, initially bearing no contribution to share, that will turn into collaborative users once their cache fills up and somebody requires the video chunks they own;
- malicious users, draining resources from the system without any purpose to collaborate: as such, their contribution to the system will always be null.

Regardless of the class the new peers belong to, the fundamental remark is that any newcomer is initially a free rider, i.e., its $u(t,x)$ is null for some time.

We can therefore conclude that $q(t,N(t)) = 0$ when $N^{'}(t) > 0$, since this term can be interpreted as the contribution of the last entered peer. This implies that $Q^{'}(t)$ in (6) can be approximated by:

$$Q'(t) \cong -\frac{N'(t)}{N(t)}Q(t) + \frac{1}{N(t)}\int_0^{N(t)} \frac{\partial}{\partial t}q(t,x)dx\,. \tag{7}$$

We observe that the previous mathematical abstraction of considering the number of peers as a continuous variable leads to the differential equation in (7), rather than to a difference equation. This will represent our starting point to examine three distinct cases.

A. Full Compensation

The first circumstance we consider is the one we term "the full compensation case", where we assume that the population of peers already in system succeeds in completely satisfying the requests of the churn.

In this favorable scenario we further make the assumption that $d(t,x) \cong d$, where d is the constant streaming rate required for the channel broadcasting. Equivalently, the download rate almost immediately hits the desired value d. The reasons behind this are twofold: as soon as the new peer joins the system, the tracker server provides it with a sufficiently wide list of potential parent peers, that the peer immediately contacts to start receiving content; besides, every chunk is fine for the peer that does not own any. Hence, its $d(t,x)$ very rapidly builds up. In Section III we will give physical evidence of this via experimental data.

Recalling (3), it is therefore possible to approximate (7) by:

$$Q'(t) \cong -\frac{N'(t)}{N(t)}Q(t) + \frac{1}{N(t)} \cdot \frac{1}{d}\int_0^{N(t)} \frac{\partial}{\partial t}u(t,x)dx\,, \tag{8}$$

Next, we observe that in the time interval $t \dots t + dt$, and by our previous assumption on $d(t, x)$, $N'(t)dt$ peers raise a global request given by $d \cdot N'(t)dt$: given that the system upload rate variation balances the global download request, we have that the following equality holds:

$$\left(\int_0^{N(t)} \frac{\partial}{\partial t} u(t, x) dx \right) dt = d \cdot N'(t) dt \,, \tag{9}$$

where the left-hand side term represents the system reaction.

Hence, we can write that:

$$N'(t) = \frac{1}{d} \cdot \int_0^{N(t)} \frac{\partial}{\partial t} u(t, x) dx. \tag{10}$$

Replacing last expression for $N^{'}(t)$ in (8), we obtain:

$$Q'(t) = -\frac{N'(t)}{N(t)} \cdot Q(t) + \frac{N'(t)}{N(t)} \,. \tag{11}$$

Given that $Q(t_0)$ and $N(t_0)$ indicate the initial $Q(t)$ and $N(t)$ values, the solution to this differential equation is

$$Q(t) = 1 - \frac{(1 - Q(t_0)) \cdot N(t_0)}{N(t)} \,, \tag{12}$$

which is amenable to a nice interpretation: as $N(t)$ grows, and in the limit goes to infinity, $Q(t)$ goes to 1.

This justifies the evidence offered by measurements on live P2P streaming systems, that demonstrate the scalability of such architectures with the number of users.

Alternatively, recalling that we have assumed that $d(t, x) = d$, $\forall x$, then the balance equality in (5) turns into

$$\int_0^{N(t)} u(t, x) dx + S = N(t) \cdot d \,; \tag{13}$$

last expression, coupled with (4), allows to derive $Q(t)$ directly from (13), obtaining

$$Q(t) = 1 - \frac{S}{d \cdot N(t)} \,, \tag{14}$$

which is equivalent to (12) and, at the same time, shows that $Q(t)$ will approach 1 from below, as $N(t)$ tends to infinity.

B. *Partial Compensation*

We next suppose that the system is partly able to cope with the requests of the new peers joining the system. And indeed, a real P2P overlay might not be able to completely answer all download requests of a massive churn that abruptly enters it. Hence, we assume that it succeeds in completely reacting and absorbing the churn only if the rate of the new entering peers remains below a certain threshold. If the threshold is trespassed, the system answers providing all its capacity, but no more than that.

The interpretation for this intermediate case is that we can reasonably expect peers already in system to provide more resources (namely, upload rate) to face new requests: however, there is a limit to the amount of bandwidth they can put into play. In other words, in a homogeneous scenario we cannot forget that

$$u(t, x) \leq u_{max} \quad \forall x \,, \tag{15}$$

and as a consequence, that system capacity is upper bounded.

We therefore introduce a function $f(N(t), N'(t))$ such that:

$$\left(\int_0^{N(t)} \frac{\partial}{\partial t} u(t, x) dx \right) dt = f(N(t), N'(t)) dt \,, \tag{16}$$

where the function $f(N(t), N'(t))$ is defined as:

$$f(N(t), N'(t)) = \begin{cases} d \cdot N'(t), & \text{if } N'(t) < \bar{p}(N(t)) \\ \\ d \cdot \bar{p}(N(t)), & \text{if } N'(t) > \bar{p}(N(t)) \end{cases} \tag{17}$$

$\bar{p}(\cdot)$ being a threshold, function of the number of peers (recall that d is the streaming rate of the broadcasted channel).

In what follows, we will make a further simplification, i.e., we will assume that $\bar{p}$ depends only on the number of peers that are already in system before the churn occurs: this corresponds to considering $\bar{p}$ constant over the interval of variability of $N(t)$.

With a more careful analysis, that is beyond the scope of this paper, it is possible to find a close relationship between $f(N(t), N'(t))$ and the growth-rate of a population of newcomers, that are not fully integrated in the well-behaved network, and thus responsible for the efficiency decrease.

Taking advantage of (16) in (7) one obtains:

$$Q'(t) = -\frac{N'(t)}{N(t)} \cdot Q(t) + \frac{1}{d} \cdot \frac{f(N(t), N'(t))}{N(t)} \,. \tag{18}$$

If we now multiply both members of (18) by $N(t)$ we have

$$N(t)Q'(t) + N'(t)Q(t) = \frac{1}{d} \cdot f(N(t), N'(t)) \,; \tag{19}$$

the left-hand side member can be seen as the derivative of a product, hence:

$$(N(t)Q(t))' = \frac{1}{d} \cdot f(N(t), N'(t))$$

$$N(t)Q(t) = \frac{1}{d} \cdot \int_0^t f(N(s), N'(s)) ds + N(t_0)Q(t_0) \tag{20}$$

so that

$$Q(t) = \frac{1}{d \cdot N(t)} \int_0^t f(N(s), N'(s)) ds + \frac{N(t_0)Q(t_0)}{N(t)} \,. \tag{21}$$

In Section III we will comment what this expression implies in terms of system efficiency $Q(t)$.

C. *Lack of reaction*

The third and last examined case foresees no reaction at all, and it may be interpreted as a limit of the previous case. As we are now supposing that the system does not counteract to the requests of new connected peers, we obtain the indication of the lowest value that $Q(t)$ can ever achieve.

In this circumstance, expression (7) can be approximated as:

$$Q'(t) \cong -\frac{N'(t)}{N(t)} Q(t) \,, \tag{22}$$

where the reaction term has been neglected.

This is a linear differential equation in $Q(t)$, that can be rewritten as:

$$\frac{Q'(t)}{Q(t)} = -(lnN(t))'$$
$$(lnQ(t))' = -(lnN(t))', \qquad (23)$$

and solved via the following passages:

$$lnQ(t) = -lnN(t) + K$$
$$lnQ(t) = ln\left(\frac{e^K}{N(t)}\right)$$
$$Q(t) = \frac{1}{N(t)}e^K. \qquad (24)$$

where constant K is given by $K = lnQ(t_0)N(t_0)$. Accordingly, $Q(t)$ is expressed by

$$Q(t) = \frac{1}{N(t)} \cdot Q(t_0)N(t_0). \qquad (25)$$

In summary, equations (12), (21) and (25) detail how system efficiency $Q(t)$ evolves as a function of $N(t)$ in the three examined cases.

A final remark is now mandatory regarding $Q(t)$ evolution after the $N(t)$ increase we are considering: the new peers will recover video content and will gradually start contributing to the system. Our current model does not (and cannot) capture such phenomenon, as it does not take into account this forcedly delayed reaction.

Nevertheless, it is valuable in outlining some asymptotic system behaviors and in delineating the regions where $Q(t)$ will evolve.

III. Experimental Measures

This Section first provides the succinct description of the P2P system that we relied upon to identify and understand the efficiency issue. It then illustrates some actual, indicative $Q(t)$ and $N(t)$ behaviors that were observed, and the estimated $Q(t)$ evolution as foreseen by the proposed model.

A. The Examined System: a Small Overlay

The system we have investigated is StreamerOne[6]: it is the first Italian P2P live video streaming platform, currently providing only a few test channels to a relatively modest floor of users. Its overlay is based on a mesh architecture, with no predefined topology. Every peer maintains a list of partners and periodically exchanges with them information about the available data, via buffer maps; it then pulls the desired video blocks from one of the peers that advertises them.

StreamerOne architecture encompasses the presence of one control server and of various streaming servers, one for each broadcasted channel. When a new peer joins the system, it receives from the control server the references to all streaming servers. Additionally, every peer receives from the control server a list of peers, not necessarily watching the same program, with which to exchange further information about the system, such as name, number of users and current efficiency of all channels.

Once the peer has selected the desired channel, it receives from the corresponding streaming server a list of 10 peers to exchange video information with. Peers are selected randomly from the set of peers watching the same channel, and the worst contributor is periodically purged from the list and replaced by a new one.

In the beginning, Streamerone broadcasted television channels with a modest rate of 160 kbit/s. Recently, its developers switched to an H.264 based transmission, with a 224 kbits/s rate, that guarantees better video quality.

As a concluding remark, the examined architecture can definitely be placed among the examples of small P2P overlays for IP-TV: the results reported hereafter evidence that even during the occurrences of very popular events the number of peers in system, $N(t)$, never exceeds a few thousands.

B. Efficiency Measures and the Simplified Model

Fig.1 reports an indicative example of the observed behavior of $N(t)$ in StreamerOne. The solid line refers to values that have been collected for a test television channel steadily gaining in popularity. The dot-dashed line that pairs $N(t)$ evolution has been obtained via a polynomial fitting of the experimental data.

Fig.2 reports the corresponding evolution of $Q(t)$ as witnessed by measurements (solid line), as well as $Q(t)$ values as predicted by (12) (dot-dashed line). $Q(t)$ analytical curve tends to 1 more slowly than the measured $Q(t)$, but this has to be expected, as: (i) the model that leads to (14) is overly simple; (ii) the experimental data are spoiled by measurement uncertainties.

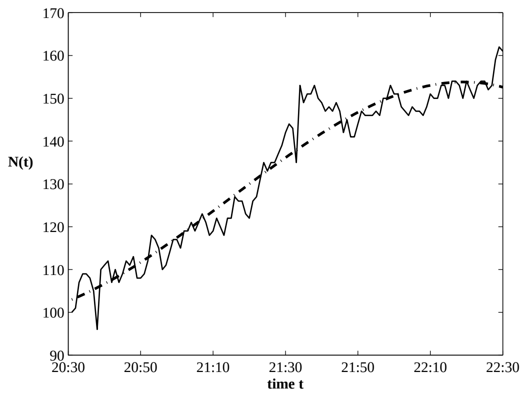


Fig. 1. An instance of $N(t)$ evolution in StreamerOne and its polynomial fitting

Fig.3 reports an instance of the evolution of $q(t, x)$ and of $u(t, x)/d$, the upload rate $u(t, x)$ normalized to the streaming rate d, for a residential peer of StreamerOne deploying an ADSL access featuring 384 kbit/s in upstream and 7 Mbit/s in downstream. These measurements refer to the very first connection minutes and have been performed during the television broadcasting of an ordinary program. By visual inspection it can be concluded that $d(t, x)$ is almost immediately equal to d, validating the assumption that $d(t, x) = d$ in a system that is not overly stressed, as asserted in Subsection II A.

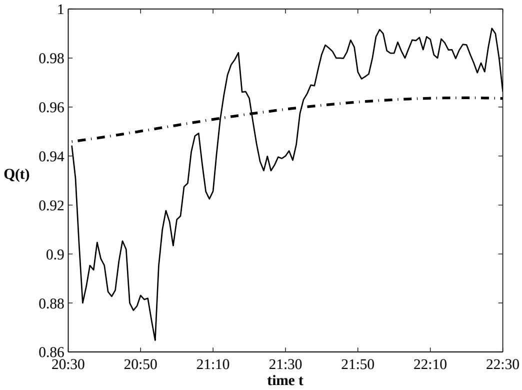


Fig. 2. $Q(t)$ corresponding to Fig.1(solid line) and $Q(t)$ as predicted by the full compensation case (dot-dashed line)

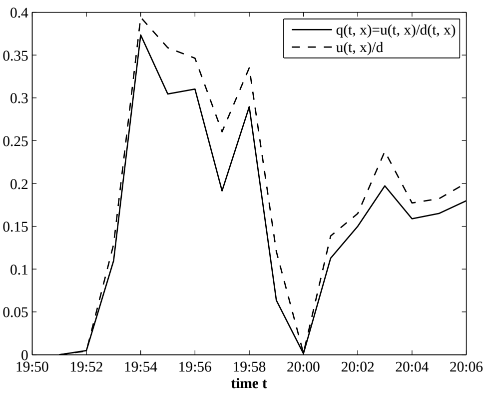
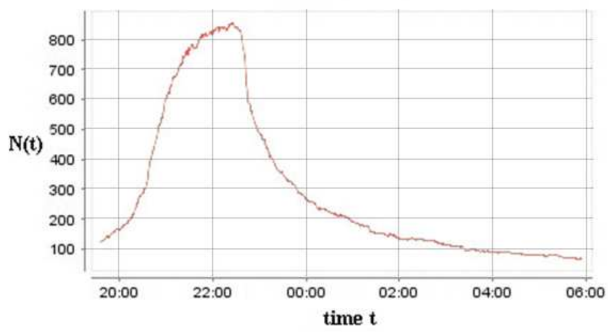
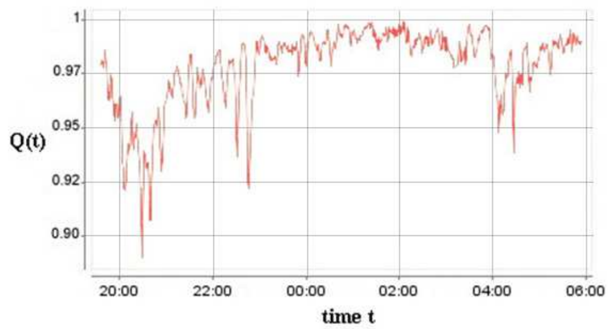


Fig. 3. $q(t, x)$ ratio and $u(t, x)/d$, upload rate normalized to the streaming rate $d = 224$ kbit/s, during the first connection minutes



(a) $N(t)$ observed behavior



(b) corresponding $Q(t)$

Fig. 4. An additional meaningful example, displaying a large churn occurrence

Incidentally, we further observe that $u(t, x)$ increases at a slower pace: its more gradual growth is explained by the fact that it takes a while before other peers in the network learn about the presence of a potential contributor; also, $u(t, x) \neq 0$ implies that some users require exactly the video chunks the peer has in its cache: no surprise that in a small system $u(t, x)/d$ is often lower than 1.

Fig.4 (a) reports an additional, interesting example of the observed $N(t)$ behavior as a function of time, and Fig.4 (b) the corresponding $Q(t)$ evolution, as monitored during the occurrence of a very popular television event. This time a significant churn is registered: as previously anticipated, it can be concluded that a sudden $N(t)$ increase reflects into an abrupt $Q(t)$ decrease. Then the system gradually recovers, as new comers start sharing content and therefore contribute to the correct system functioning.

In our analysis we reproduce such sharp $N(t)$ increase via an approximated step function, given by:

$$N(t) = \begin{cases} N(t_0), & \text{if } t_0 < t < t_1 \\ N(t_0) + \frac{(N(t_1) - N(t_0)) \cdot (t - t_1)}{\epsilon}, & \text{if } t_1 < t < t_1 + \epsilon \\ N(t_1), & \text{if } t_1 + \epsilon < t < t_{max} \end{cases} \quad (26)$$

Fig.5 reports the approximation for t_0 equal to 19:00, t_1 to 21:00, $N(t_0) = 120$, $N(t_1) = 850$ and $\epsilon = 0.25$.

Fig.6 displays its $Q(t)$ counterparts in the three cases we have previously commented.

In detail, the dot line refers to the ideal case of perfect compensation: we observe that in this circumstance $Q(t)$ would steadily increase and from $Q(t_0) \simeq 0.98$ it would swiftly reach unity.

The dashed curve refers to the case of partial compensation, where the system copes with the requests up to a threshold. As previously underlined, the current model cannot capture $Q(t)$ recovery expected after the initial decrease due to the churn entering the system; nevertheless, it can seize the lowest value that $Q(t)$ will hit (0.9, as inferred from the measurements of Fig.4, for $\bar{p} = 2590$) and the time it will take to reach such value (a few minutes).

Finally, the solid line represents $Q(t)$ in the pessimistic circumstance where the system lacks reaction. This is $Q(t)$ evolution that would be observed if the system were not able to cope with the new peers, given its overall resources (upload rates) had been already exhausted. It corresponds to an observed value $Q(t) = 0.14$ after the churn occurrence, which is equal to $1 - \frac{(N(t_1) - N(t_0))}{N(t_1)} = 0.14$, i.e., to 1 minus the fraction of newcomers over the total peer population. In other words, only the old peers already in system keep receiving

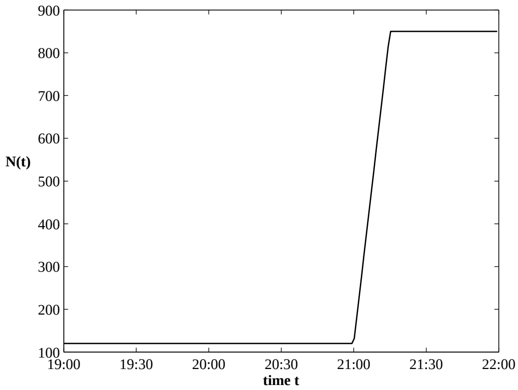


Fig. 5. Number of peers $N(t)$ via the step function approximation

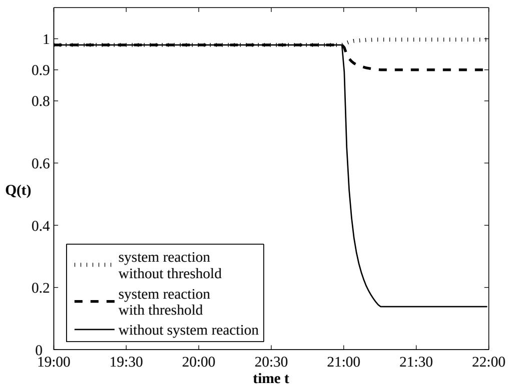


Fig. 6. $Q(t)$ evolution corresponding to $N(t)$ as reported in Fig.5

the video streams (summing up to 14% of the population); newcomers are locked out (86% of the population, in this example).

IV. Conclusions

This paper has presented an analytical model to understand how network efficiency, and therefore quality in a P2P system for video broadcasting, is affected by the number of peers in system. The main goals were to find an analytical justification to the scalability of P2P architectures and also to understand to what extent system efficiency varies after a sudden increase of connected peers. Although the achieved results have highlighted some important behavioral aspects of the examined system, an additional refinement of the analytical model is mandatory, to capture the dynamics of efficiency recovery after the sharp, positive peer churn.

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Appendix

Given the function $g : I \times J \rightarrow \Re$ is continuous and has partial derivative $\frac{\partial}{\partial x} g(x,y)$ in $I \times J$ and α, $\beta : I \rightarrow J$ are continuous functions with continuous first derivative, then the derivative of $f(x)$, with

$$f(x) = \int_{\alpha(x)}^{\beta(x)} g(x,y)dy$$

is

$$f'(x) = \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} g(x,y)dy - \alpha'(x)g(x,\alpha(x)) + \beta'(x)g(x,\beta(x)).$$

Therefore, the derivative of (4) is:

$$Q'(t) = \frac{N(t)}{N(t)^2} \cdot \left(\int_0^{N(t)} \frac{\partial}{\partial t} q(t,x)dx + N'(t) \cdot q(t,N(t)) \right) + \\ - \frac{N'(t)}{N(t)^2} \cdot \int_0^{N(t)} q(t,x)dx = \\ = \frac{1}{N(t)} \cdot \int_0^{N(t)} \frac{\partial}{\partial t} q(t,x)dx + \frac{N'(t)}{N(t)} \cdot q(t,N(t)) + \\ - \frac{N'(t)}{N(t)} \cdot \frac{\int_0^{N(t)} q(t,x)dx}{N(t)}.$$

Recalling $Q(t)$ definition as given by (4), $Q'(t)$ can also be written as:

$$Q'(t) = -\frac{N'(t)}{N(t)} Q(t) + \frac{1}{N(t)} \int_0^{N(t)} \frac{\partial}{\partial t} q(t,x)dx + \\ + \frac{N'(t)}{N(t)} q(t,N(t)) , \quad (27)$$

as it appears in (6).