Modeling and Simulation of Static and Coulomb Friction in a Class of Automotive Systems

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Abstract - Static and Coulomb friction are extensively used in automotive mechanical systems to control the synchronization between two shafts or two axles. Clutches, gearboxes, limited-slip differentials and brakes are some examples.

This paper proposes a method for the efficient simulation of a wide class of automotive mechanical systems with static and Coulomb friction phenomena. The modeling approach is based on the port-Hamiltonian systems and the computation of the friction forces requires only the zero crossing detection. A slight approximation allows faster and sufficiently accurate simulations even without an accurate zero crossing detection.

The proposed approach have been used to simulate the behavior of a complex gearbox provided by some high level farm tractors.

Keywords: Static Friction, Coulomb Friction, Automotive, Modeling, Simulation.

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1 Introduction

Static and Coulomb friction are common phenomena in several automotive mechanical systems. The static and Coulomb friction forces are usually considered as undesired dissipative phenomena in mechanical systems. However, the static and Coulomb friction are widely used in automotive mechanical systems to control the synchronization between two shafts or two axles. The simplest example is given by the clutch (see Fig. 1), this example gives some insight about the issues of modeling and simulating the systems with static and Coulomb friction. Several other examples are available and will be introduced soon. When the clutch is slipping, the two axles J_1 and J_2 move almost independently (one respect to the other) under the action of the torques τ_1 and τ_2, and the Coulomb friction $\tau_{cf} = k_{1,2}\,\mathrm{sign}(\omega_1 - \omega_2)$ is exchanged between them. The Coulomb friction aims to reduce the relative speed $\omega_1 - \omega_2$ between the two axles:

$$
\begin{aligned}
J_1\,\dot\omega_1 &= \tau_1 - \tau_{cf} = \tau_1 - k_{1,2}\,\mathrm{sign}(\omega_1 - \omega_2)\\
J_2\,\dot\omega_2 &= \tau_{cf} - \tau_2 = k_{1,2}\,\mathrm{sign}(\omega_1 - \omega_2) - \tau_2
\end{aligned}
$$

When the clutch is engaged, the two axles J_1 and J_2 rotate together and the relative speed $\omega_1 - \omega_2$ is zero. In this case the two axles are kept together thanks to the static friction due to the clutch. The torque $\lambda_{1,2}$ necessary to keep the two axles rotating together

$$
\begin{aligned}
J_1\,\dot\omega_1 &= \tau_1 - \lambda_{1,2}\\
J_2\,\dot\omega_2 &= \lambda_{1,2} - \tau_2\\
\omega_1 &= \omega_2
\end{aligned}
$$

can be computed by imposing that the derivative of the relative speed is equal to zero when $\omega_1 = \omega_2$:

$$
\dot\omega_1 - \dot\omega_2 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = \frac{J_2\,\tau_1 + J_1\,\tau_2}{J_1 + J_2}
$$

The static friction τ_{sf} aims to keep the two axles together, but it must be lower than the static friction maximum amplitude $K_{1,2} > 0$, then:

$$
\tau_{sf} = \begin{cases} \lambda_{1,2} & \text{if} \quad |\lambda_{1,2}| \le K_{1,2} \\ K_{1,2}\,\mathrm{sign}(\lambda_{1,2}) & \text{otherwise} \end{cases}
$$

When the relative speed is zero the dynamics of the clutch system is:

$$
\begin{aligned}
J_1\,\dot\omega_1 &= \tau_1 - \tau_{sf}\\
J_2\,\dot\omega_2 &= \tau_{sf} - \tau_2
\end{aligned}
$$

If $|\lambda_{1,2}| \le K_{1,2}$ and $\omega_1 - \omega_2 = 0$, the dynamics of the clutch system can be described by a simpler first order model:

$$
\begin{aligned}
(J_1 + J_2)\,\dot\omega_1 &= \tau_1 - \tau_2\\
\omega_2 &= \omega_1
\end{aligned}
\tag{1}
$$

The dynamics of the clutch system shown in Fig. 1 can be finally expressed by:

$$
\begin{aligned}
J_1\,\dot{\omega}_1 &= \tau_1 - \mu_{1,2} \\
J_2\,\dot{\omega}_2 &= \mu_{1,2} - \tau_2 \\
\mu_{1,2} &= \begin{cases} k_{1,2}\,\mathrm{sign}(\omega_1-\omega_2) & \text{if} \quad \omega_1-\omega_2 \neq 0 \\ \lambda_{1,2} & \text{if} \quad |\lambda_{1,2}| \leq K_{1,2} \ \text{ and } \ \omega_1-\omega_2 = 0 \\ K_{1,2}\,\mathrm{sign}(\lambda_{1,2}) & \text{if} \quad |\lambda_{1,2}| > K_{1,2} \ \text{ and } \ \omega_1-\omega_2 = 0 \end{cases} \\
\lambda_{1,2} &= \frac{J_2\,\tau_1 + J_1\,\tau_2}{J_1+J_2}
\end{aligned} \qquad (2)
$$

where $k_{1,2}$ is the amplitude of the Coulomb friction and $K_{1,2}$ denotes the static friction maximum amplitude. Usually $K_{1,2} \geq k_{1,2}$ and both depend on the external normal force that compresses the clutch disks. Moreover, $k_{1,2}$ may be dependent on the relative speed (the Stribeck effect, see [Armstrong-Hélouvry et al. 1994]).

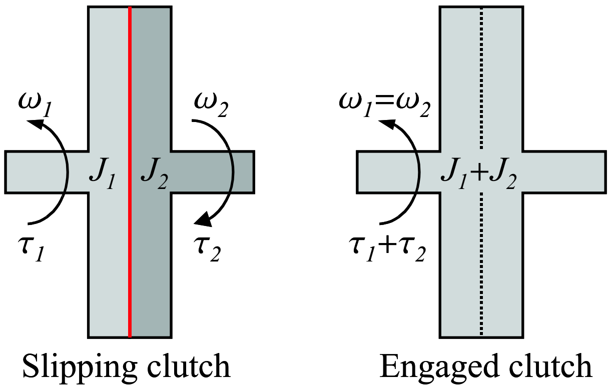

Figure 1: Description of the two operating states of a dry clutch.

The static and Coulomb Friction force $\mu_{1,2}$ aims to reduce the relative speed $\omega_1 - \omega_2$ (Coulomb friction) or to keep the two axles at the same angular speed when $\omega_1 = \omega_2$ (static friction). For this reason the static and Coulomb friction are widely used in the automotive mechanical systems to control the synchronization between two shafts or two axles by means of clutches or similar devices. Here is a brief list of applications coming from different car industries and already available at least on known prototypes. The dry clutch is used on almost all commercial cars and it has been just described. The dual clutch transmission system works in a similar way by using two clutches in order to avoid the torque interruption during the gear shift. The Coulomb friction is also used within the synchronizers of a traditional gearbox to synchronize the speeds of the primary and the propeller shafts to allow the engagement of a new gear. High-level farm tractors are equipped with the *Powershift* gearbox, its main feature is to allow a full automatic gear shift without traction torque interruption. It is briefly described in Section 5, in this case the friction forces are used to synchronize and to keep engaged the gears. Several examples of the use of clutches (and therefore friction) within automotive transmissions can be found in [Jurghen 2000]. The limited slip differentials [Morselli et al. 2005] allow to modify the torque distribution on the driving wheels, its important effects on the vehicle dynamics are well known [Wright 2000]. Finally, the drum and the disk braking systems operate by using the Coulomb friction.

The correct and fast simulation of the clutch system (2) and, more generally, of systems with static and Coulomb friction is a well known issue. The problem is essentially the accurate simulation and computation of the term $\mu_{1,2}$ in (2) for each clutch in the system. The simplest simulation models do not compute the term $\lambda_{1,2}$ in (2) and $\mu_{1,2}$ is given by the sign function only. When the difference $(\omega_1 - \omega_2)$ becomes zero, the term $k_{1,2}\,\mathrm{sign}(\omega_1 - \omega_2)$ starts switching between $\pm k_{1,2}$, and to achieve reliable simulation results the simulation step size must be as small as possible (the simulation gets stuck). The problem of the proper simulation of Coulomb friction is a well studied subject and various solutions are proposed in literature. A survey of friction models and simulation techniques concerning the Coulomb friction is given in [Armstrong-Hélouvry et al. 1994]. Some energy based accurate solutions can be found in [Breedveld 2000].

From a system theory point of view, a system with static and Coulomb friction is a so called *Variable Dynamic Dimension System* (VDDS) (see [Zanasi et al. 2001]). Due to strong nonlinearities (i.e. the sign function), a system belongings to this class changes its dynamic dimension (i.e. system order) during normal operations. From a mathematical point of view, the system order reduction means that some state variable derivatives are constrained to remain zero until some dynamical conditions are satisfied. For the clutch if $|\lambda_{1,2}| \leq K_{1,2}$ the two axles J_1 and J_2 are constrained to rotate at the same speed: $\omega_1 = \omega_2$ and the dynamic dimension is reduced by 1. This condition corresponds to the model (1). The hybrid system approach [v.d.Schaft and Schumacher 2000] essentially consists in commuting between all the possible sub-models. This approach can be followed for systems with relative small complexity as the clutch [Garofalo et al. 2001], but it is not suitable when the number of sub-models is high, as for the system described in Section 5.

Concerning the automotive mechanical systems, following the concept of VDDS and by using a proper state space congruent transformation it is possible to obtain a n dimensional state vector whose first variable represents the main dynamic of the system (i.e. when all the clutches are fully engaged), the remaining variables take into account the dynamics of the relative speeds between the axles when the corresponding clutches are slipping. By this way, detecting the zero crossing of the relative speeds, it is possible to simulate accurately many automotive systems with clutches: when a clutch is fully engaged, the corresponding state variable is kept to zero. This solution has been successfully proposed for the clutch with torque damper [Zanasi et al. 2001], for the gearbox [Zanasi et al. 2002] and for the controlled limited-slip differentials [Zanasi et al. 2003] and [Morselli et al. 2005]. These solutions differ from the preceding ones thanks to the state space transformation that allows to deal with the Coulomb friction without the need of models switching, pre-slip velocities or hybrid resistive ports. The simulation results are exact and the simulations only require the detection of the zero relative velocities. Moreover, the solutions presented in [Armstrong-Hélouvry et al. 1994] and [Breedveld 2000] are not applicable directly to some automotive mechanical systems. However these solutions are suited for single mechanical systems since the state transformations depend on the mechanical framework and differ from one system to another, moreover this way is not applicable to the Powershift gearbox presented in Section 5.

The solution proposed in this paper requires the same computational effort than the previous (zero crossing detection, same number of integrators in the model, computation of the friction forces), however it is applicable to a wide class of automotive systems, especially all the systems previously mentioned. The proposed approach is based on a *port-Hamiltonian* modeling of the mechanical system and on the computation of the static friction forces or torques as they were constraint forces or torques. The proposed approach ensures sufficiently accurate results for most applications even if the zero crossing detection is not exact.

The paper is organized as follows. The Section 2 gives a brief introduction on port-Hamiltonian systems. The class of automotive systems considered in the paper is described by means of the port-Hamiltonian framework in Section 3. The computation of the static and Coulomb friction forces and the simulation of the models is described in Section 4. Finally, Section 5 presents some simulation results referred to the application of the proposed approach to the Powershift gearbox.

2 A Brief Introduction on Port-Hamiltonian Systems

The hamiltonian framework is a powerful means to model dynamic systems. A brief recall of the definitions given in [v.d.Schaft and Maschke 1994] and in in [v.d.Schaft and Maschke 1994] and in [v.d.Schaft 2000] is given herein for reader convenience.

The standard *Euler-Lagrange equations* are given as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau \tag{3}$$

where $q = (q_1, \ldots, q_n)^T$ are generalized configuration coordinates for the system with n degrees of freedom, $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)^T$ are the generalized velocities and $\tau = (\tau_1, \ldots, \tau_n)^T$ is the vector of the generalized forces acting on the system. The *Lagrangian* $L(q,\dot{q})$ equals the difference between the kinetic energy $K(q,\dot{q})$ and the potential energy $V(q)$:

$$L(q,\dot{q}) = K(q,\dot{q}) - V(q)$$

The partial derivatives $\frac{\partial L}{\partial \dot{q}}$ and $\frac{\partial L}{\partial q}$ are column vector. The Lagrangian function $L(q,\dot{q})$ in standard mechanical systems is of the form:

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T \mathbf{M}(q)\,\dot{q} - V(q) \tag{4}$$

where the $n \times n$ inertia matrix $M(q)$ (generalized mass) is symmetric and positive definite for all q. The vector of generalized *momenta* $p = (p_1, \ldots, p_n)^T$ is defined for any Lagrangian as:

$$p = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$$

For a standard mechanical systems with Lagrangian (4), the generalized momenta are of the form:

$$p = \mathbf{M}(q)\,\dot{q} \tag{5}$$

By using the state vector $(q_1, \ldots, q_n, p_1, \ldots, p_n)^T$ the n second-order equations (3) transforms into a $2n$ first-order equations called *Hamiltonian equations* of motion:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \tau \end{aligned} \tag{6}$$

where the *Hamiltonian* $H(q,p)$ is the total energy of the system:

$$H(q,p) = K(q,p) + V(q)$$

System (6) is an example of *Hamiltonian system* which more generally is given in the following form:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \mathbf{B}(q)u \\ y &= \mathbf{B}^T(q)\frac{\partial H}{\partial p}(q,p) = \mathbf{B}^T(q)\dot{q} \end{aligned} \tag{7}$$

where $B(q) \in \mathbb{R}^{n\times m}$ is the input force matrix, $B(q)\,u$ denotes the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ are the outputs. The power flowing into the system is:

$$\frac{dH}{dt}(q(t), p(t)) = u^T(t)\,y(t) \tag{8}$$

therefore the pair (u, y) represents a *power-port* between the Hamiltonian system and the external world.

The Hamiltonian framework is suitable to compute and include classical constraints expressed in local coordinates as:

$$\mathbf{A}^T(q)\,\dot{q} = 0 \tag{9}$$

The Hamiltonian system (7) with classical constraints takes the form:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + \mathbf{B}(q)\,u + \mathbf{A}(q)\,\lambda \\ y &= \mathbf{B}^T(q)\frac{\partial H}{\partial p}(q,p) = \mathbf{B}^T(q)\,\dot{q} \\ 0 &= \mathbf{A}^T(q)\,\dot{q} \end{aligned} \tag{10}$$

where $\mathbf{A}(q)\,\lambda$ denotes the constraint forces. Vector λ can be usually computed by differentiating the constraints (9) as described in [v.d.Schaft and Maschke 1994]. Note that the constraint forces do not dissipate nor generate energy in the system, therefore the energy balance is still the same as in (8).

3 Model of the mechanical system

The class of automotive mechanical systems considered in this paper consists in n axles connected by gears and clutches. A typical example is the gearbox shown in Fig. 3 and described in Section 5. All the mechanical system previously cited belong to this class.

The generalized positions are the angular positions θ_i for $i = 1 \dots n$, the generalized momenta are $p_i = J_i \dot{\theta}_i = J_i \omega_i$ for $i = 1 \dots n$, where J_i denotes the axles inertia and ω_i is the corresponding angular velocity. Let $p = [p_1, \dots, p_n]^T$ and $\omega = [\omega_1, \dots, \omega_n]^T$. The Hamiltonian is then given by the sum of the kinetic energy of all the axles:

$$H(p) = \frac{1}{2} p^T \mathbf{M}^{-1} p = \frac{1}{2} \omega^T \mathbf{M} \omega = H(\omega) \tag{11}$$

where $\mathbf{M}$ is the constant diagonal inertia matrix with elements $J_1, \dots, J_n$.

Each gear couples the velocity of two axles and can have a clutch to force the synchronization and the full engagement of the gear. Each clutch works on the relative speed imposed by the gear. All the gears ratios can be described by a matrix $\mathbf{R} \in \mathbb{R}^{g \times n}$ with the following framework:

$$\mathbf{R} = \begin{bmatrix} r_{1,1} & r_{1,2} & 0 & \dots & 0 & 0 \\ r_{2,1} & r_{2,2} & 0 & \dots & 0 & 0 \\ 0 & r_{3,2} & r_{3,2} & \dots & 0 & 0 \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & r_{g-1,n-1} & r_{g-1,n} \\ 0 & 0 & 0 & \dots & r_{g,n-1} & r_{g,n} \end{bmatrix} \tag{12}$$

Each row R_i of matrix $\mathbf{R}$ represents a gear, it has two and only two non zero terms that represent the mean radius of the two disks coupled by the gear, see Fig. 3. All the gear ratios are different, therefore the rows of $\mathbf{R}$ taken two by two are not proportional. When the i^{th} gear is fully engaged, the constraint $R_i \omega = 0$ is satisfied. For a direct gear both non zero terms in the corresponding row are positive, therefore the angular velocities of the two axles have opposite sign when the gear is engaged. For a reverse gear the second non zero term in the corresponding row is negative.

Let $\mu = [\mu_1, \dots, \mu_g]^T$ denote the tangential forces (to be computed) exchanged within the gears. The analytical description of the class of automotive systems considered in this paper is given by the following Hamiltonian system:

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p}(p) = \mathbf{M}^{-1} p = \omega \\ \dot{p} &= \mathbf{M} \dot{\omega} = \mathbf{B}(\theta) \tau + \mathbf{R}^T \mu \\ y &= \mathbf{B}^T(\theta) \frac{\partial H}{\partial p}(p) = \mathbf{B}^T(\theta) \omega \\ \delta &= \mathbf{R} \omega \end{aligned} \tag{13}$$

where $\delta = [\delta_1, \dots, \delta_g]$ is the vector of the peripheral relative speeds across the gear wheels, if δ_i is kept to zero the clutch i is engaged and therefore also the gear i is engaged. The power-port (τ, y) allows to connect the system (13) to other mechanical systems (i.e. the engine, the wheels,...) or to include viscous friction forces or other loads. The static and Coulomb friction forces are gathered in the force vector $\mu(\delta,\tau)$. The elements of matrix $\mathbf{R}$ are linear radii therefore δ represents linear speeds and consequently μ is a vector of forces. These friction forces can be real forces or the equivalent effect due to friction torques, as usually happens for the clutches (see also the example in Section 5). Another choice is possible with the same analytical results: if the two non zero terms of each row of $\mathbf{R}$ were 1 and the gear ratio then μ would be the vector of the friction torques and δ would represent the angular relative speeds.

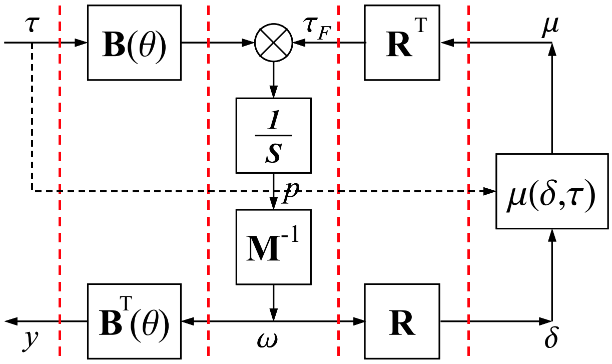

Figure 2: Power-oriented scheme for the simulation of system (13).

The dynamic system defined by equations (13) can be simulated by using the block scheme shown in Fig. 2. The basic idea of this scheme is to use the power interaction between subsystems as basic concept for modeling, see [Karnopp et al. 2000] and [Zanasi 1991]. The dashed lines represent the power ports between the subsystems and the inner product of the two vectorial variables involved in each dashed line of the graph has the physical meaning of the power flowing through the section. The symbol $\otimes$ means the sum of the input variables. The presence of the integrators (integral causality) and the absence of derivators (differential causality), see [Breedveld 2004], allow the reliable simulation of this scheme using every computer simulation. The key issue is the computation of the (equivalent) friction forces μ in order to allow the simulation of both slipping and engaged clutches without switching between different models. Unfortunately the friction forces $\mu(\delta,\tau)$ depend on both ω and τ so the matrix $\mathbf{R}$ and the pair (τ_F,ω) define a dissipative port modulated by τ or an *hybrid resistive port* as defined in [Breedveld 2000].

4 Computation of the static and Coulomb friction

Let $k(F_N,\delta) = [k_1(F_{N1},\delta_1),\ldots,k_g(F_{Ng},\delta_g)]^T$ denote the vector of the amplitudes of the Coulomb friction of all the clutches, all the components of k are positive or zero. The amplitude k_i mainly depends on the normal force F_{Ni} that compresses the clutch disks and may depend on the disks relative speed δ_i (Stribeck effect). Also thermal effects can be easily taken into account. Let $\mathbf{k}(F_N,\delta) = \operatorname{diag}[k_1(F_{N1},\delta_1),\ldots,k_g(F_{Ng},\delta_g)]$ be the diagonal matrix with elements $k_1(F_{N1},\delta_1)$,..., $k_g(F_{Ng},\delta_g)$. Let $K(F_N) = [K_1(F_{N1}),\ldots,K_g(F_{Ng})]^T$ denote the vector of the non negative amplitudes of the static friction, it is assumed that $K_i(F_{Ni}) \geq k_i(F_{Ni},0)$. Let $\mathbf{K}(F_N) = \operatorname{diag}[K_1(F_{N1}),\ldots,K_g(F_{Ng})]$ be the diagonal matrix with the amplitudes of the static friction.

Let ν denote the set of indexes j that correspond to $K_j > 0$ and zero relative speeds δ_j. Let σ denote the set of remaining indexes i that correspond to the non zero relative speeds δ_i or to zero relative speed with $K_i = 0$:

$$\nu = \{j : \delta_j = 0 \ \wedge \ K_j > 0\} \qquad \sigma = \{i : (\delta_i \neq 0) \ \vee \ (\delta_i = 0 \ \wedge \ K_i = 0)\} \tag{14}$$

The two sets are complementary: $\sigma \cap \nu = \emptyset$ and $\sigma \cup \nu = \{1, \ldots, g\}$. Let $\mathbf{R}_\nu$ and $\mathbf{R}_\sigma$ denote the matrix composed by the rows of $\mathbf{R}$ corresponding to the set ν and σ respectively. Finally, let ω_ν [ω_σ, μ_ν,...] denote the components of the vector ω [ω, μ,...] corresponding to the set ν [σ, ν,...]. Following these definitions $\delta_\sigma = \mathbf{R}_\sigma \, \omega$ and $\delta_\nu = \mathbf{R}_\nu \, \omega = 0$.

The computation of the Coulomb friction forces μ_σ corresponding to the slipping clutches (non zero relative speeds) is straightforward:

$$\mu_\sigma = -\mathbf{k}_\sigma(F_{N\sigma}, \delta_\sigma) \, \mathrm{sign}(\delta_\sigma) \tag{15}$$

indeed the power flowing through the power-port (μ, δ) or (τ_F, ω) due to (15) is given by:

$$\omega^T \, \tau_F = \delta_\sigma^T \, \mu_\sigma \; = -\delta_\sigma^T \, \mathbf{k}_\sigma(F_{N\sigma}, \delta_\sigma) \, \mathrm{sign}(\delta_\sigma) = -|\delta_\sigma|^T \, k_\sigma(F_{N\sigma}, \delta_\sigma) \leq 0$$

therefore the Coulomb friction forces (15) dissipate the kinetic energy of the axles ($|\delta_\sigma|$ is intended as the vector of the absolute values of the components and not the norm of the vector δ_σ). If $\tau = 0$, the relative speed δ_i for $i \in \sigma$ tends to zero if $k_i > 0$ (the axles tend to be synchronized).

The computation of the static friction forces corresponding to the (at least temporarily) engaged clutches (zero relative speeds with non zero static friction amplitude K) is more involved. Unfortunately their values depend also on τ and on μ_σ. The static friction forces aim to keep at zero the relative speeds, therefore they can be seen as constraint forces between the axles than aim to keep the constraint $\mathbf{R}_\nu \, \omega = 0$. Differentiating this constraint, it is possible to find the virtual forces λ that allow to satisfy the constraint:

$$\mathbf{R}_\nu \, \dot{\omega} = \mathbf{R}_\nu \mathbf{M}^{-1} \left[\mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma\right] + \mathbf{R}_\nu \, \mathbf{M}^{-1} \, \mathbf{R}_\nu^T \, \lambda = 0 \tag{16}$$

This equation can be solved for λ if $\det(\mathbf{R}_\nu \, \mathbf{M}^{-1} \, \mathbf{R}_\nu^T) \neq 0$. Since $\mathbf{M}^{-1}$ is positive defined, this condition is satisfied only if $\mathbf{R}_\nu$ is a full rank matrix. Taking into account the properties of the matrix $\mathbf{R}_\nu$, the following cases are possible (see also the example in Section 5):

1) $\mathbf{R}_\nu$ is a full rank matrix with at most $n-1$ rows. The constraint forces λ can be computed form (16). The number of constraints is lower than the system dimension, therefore the constraint $\mathbf{R}_\nu \, \omega = 0$ allows solutions with non zero speeds. If $\mathbf{R}_\nu$ has exactly $n-1$ rows and if the static friction was given by λ, the system has only one degree of freedom and all the axles are constrained to rotate with constant speed ratios.

2) $\mathbf{R}_\nu$ is a full rank $n \times n$ matrix. The constraint $\mathbf{R}_\nu \, \omega = 0$ allows only the trivial solution $\omega = 0$. If the static friction was given by λ computed form (16), the system does not have any degree of freedom an the axles are all stuck.

3) $\mathbf{R}_\nu$ is not a full rank matrix. There are infinite solutions for λ in (16) and the system may be over constrained with two or more axles stuck together.

From the application point of view only cases 1 and 2 are interesting therefore, from now on, we assume that $\mathbf{R}_\nu$ is a full rank matrix. The condition 3 is easily avoided by a proper control of the clutches. Computing λ from (16):

$$\lambda = -(\mathbf{R}_\nu \, \mathbf{M}^{-1} \, \mathbf{R}_\nu^T)^{-1} \, \mathbf{R}_\nu \, \mathbf{M}^{-1} \, [\mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma] \tag{17}$$

If the virtual constraint forces λ satisfy $|\lambda| \leq K_\nu(F_{N\nu})$ (where $|\lambda|$ means the absolute value of each component and $|\lambda| \leq K_\nu(F_{N\nu})$ is evaluated component by component), the static friction amplitude necessary to keep engaged the clutches is lower than the maximum amplitude $K_\nu(F_{N\nu})$ therefore the static friction is exactly equal to λ and the dynamics of the system is given by (13) where the second equation becomes:

$$\dot{p} = \mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma + \mathbf{R}_\nu^T \, \lambda$$

If the condition $|\lambda| \leq K_\nu(F_{N\nu})$ is not satisfied for some component of λ, the computation of the static friction amplitude becomes more complex. Let ν_c and ν_s be a partition of the set ν such that $\nu_c \cup \nu_s = \nu$ and $\nu_c \cap \nu_s = \emptyset$. This partition corresponds to a partition of the rows of the matrix $\mathbf{R}_\nu$ [vector λ, K_ν,...] into two matrices $\mathbf{R}_{\nu c}$ [λ_c, $K_{\nu c}$,...] and $\mathbf{R}_{\nu s}$ [λ_s, $K_{\nu c}$,...]. Using the new symbols the dynamics of the system is given by (13) where the second equation becomes:

$$\dot{p} = \mathbf{M} \, \dot{\omega} = \mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma + \mathbf{R}_{\nu c}^T \, \lambda_c + \mathbf{R}_{\nu s}^T \, \lambda_s$$

where μ_σ is given by (15). The forces λ have been split in two: λ_c and λ_s. Vector λ_c represents the constraint forces that do not exceed the maximum static friction amplitude, the forces λ_s denote the constraint forces that assume the maximum (or minimum) static friction amplitude.

The partitions ν_c and ν_s and the two set of friction forces λ_c and λ_s can be computed by solving the following system:

$$\left\{ \begin{array}{rcl} \lambda_c & = & -(\mathbf{R}_{\nu c} \, \mathbf{M}^{-1} \, \mathbf{R}_{\nu c}^T)^{-1} \, \mathbf{R}_{\nu c} \, \mathbf{M}^{-1} \, [\mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma + \mathbf{R}_{\nu s}^T \, \lambda_s] \\ |\lambda_c| & \leq & K_{\nu c} \\ \lambda_s & = & \mathbf{K}_{\nu s} \, \mathrm{sign} \left(-(\mathbf{R}_{\nu s} \, \mathbf{M}^{-1} \, \mathbf{R}_{\nu s}^T)^{-1} \, \mathbf{R}_{\nu s} \, \mathbf{M}^{-1} \, [\mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma + \mathbf{R}_{\nu c}^T \, \lambda_c] \right) \\ K_{\nu s} & < & \left| (\mathbf{R}_{\nu s} \, \mathbf{M}^{-1} \, \mathbf{R}_{\nu s}^T)^{-1} \, \mathbf{R}_{\nu s} \, \mathbf{M}^{-1} \, [\mathbf{B}(\theta) \, \tau + \mathbf{R}_\sigma^T \, \mu_\sigma + \mathbf{R}_{\nu c}^T \, \lambda_c] \right| \end{array} \right. \tag{18}$$

The inequalities are to be considered component by component. The first equation is exactly similar to (16), it computes the static friction forces able to keep the relative speeds $\delta_{\nu c}$ equal to zero, the amplitude of these forces must be lower than the maximum static friction amplitude $K_{\nu c}$ (first inequality). The argument of the sign function in the third equation gives the forces necessary to keep the relative speeds $\delta_{\nu s}$ equal to zero, these constraint forces exceed the maximum amplitude (last inequality) and therefore they are saturated by the sign function to obtain the static friction forces.

Unfortunately the solution of (18) can be obtained by a trial and error procedure that tries all the possible solutions. Moreover in the general case it is not ensured that one and only one solution exists as described in [Armstrong-Hélouvry et al. 1994]. However from an application perspective the problem is much more simple. For the automotive mechanical systems considered in this paper it is unlikely that two clutches reach the maximum static friction amplitude exactly at the same time instant, this is even less probable for more than two clutches. As shown later, the solution of (18) when only one constraint exceeds the maximum static friction is quite simple, therefore solving (18) requires a time consuming computation rarely.

By decomposing the friction forces μ in (13) into the Coulomb friction forces μ_σ, the non-saturated static friction forces $\lambda_{\nu c}$ and the saturated static friction forces $\lambda_{\nu s}$, the dynamics of the system (13) is described by:

$$\begin{aligned} \dot{\theta} &= \omega \\ \mathbf{M}\,\dot{\omega} &= \mathbf{B}(\theta)\,\tau + \mathbf{R}_\sigma^T\,\mu_\sigma + \mathbf{R}_{\nu c}^T\,\lambda_c + \mathbf{R}_{\nu s}^T\,\lambda_s \\ y &= \mathbf{B}^T(\theta)\,\omega \\ \delta &= \mathbf{R}\,\omega \end{aligned} \tag{19}$$

To simulate the dynamics of the system (19), with the hypothesis of an unique solution of (18), the proposed procedure is the following:

1) Check the set of zero relative speeds (zero crossing detection). Compute the two sets σ and ν according to (14).

2) Compute the Coulomb friction forces μ_σ by (15).

3) Compute the virtual constraints λ by (17) with μ_σ given by (15). If $|\lambda| \leq K_\nu(F_{N\nu})$ then $\lambda_c = \lambda$, $\nu_c = \nu$, $\nu_s = \emptyset$ and $\lambda_s = \emptyset$ is a solution of (18). The computation of the static friction forces is completed and system (19) is simulated with $\lambda_c = \lambda$ and $\lambda_s = 0$.

4) If only one component of λ exceeds the maximum static friction amplitude, let $i \in \nu$ be the corresponding index of the component that exceeds the constraint, let $\overline{\lambda}$ be the component that exceeds the maximum static friction amplitude: $|\overline{\lambda}| > K_i$. Compute λ_c from the first equation of (18) with $\nu_s = \{i\}$, $\nu_c = \nu - \{i\}$, $\lambda_s = K_{\nu s}\,\mathrm{sign}(\overline{\lambda})$, μ_σ given by (15). If $|\lambda_c| < K_{\nu c}$ the solution of (18) for λ_c and λ_s is found. Namely the exceeding constraint is saturated and the remaining are computed again: if they satisfy the boundary the solution of (18) is found.

5) More than one component of λ exceed the maximum amplitude or the previous step did not find an immediate solution. In this case, the solution of (18) have to be found by a trial and error procedure that ends as soon as a solution is found.

This procedure checks first if the most common case happens and therefore the computationally demanding solution of (18) is not necessary in most of the cases. For this reason the proposed solution seems to be quite efficient.

If the hypothesis of an unique solution of (18) is not likely, the solution(s) of (18) must be found at each simulation step. If no solution is found or if more than one solution are found, it is necessary to understand by a deep analysis of the mechanical behaviour which are the right values of the friction forces, the answer to this question depends strongly on the features of the system and varies from one system to another.

It is interesting to note some robustness properties of the proposed approach. If the zero relative speeds are not accurately detected (namely $|\delta_\nu| < \varepsilon$ is considered to be zero) the computation of the friction forces by (18) ensures $\dot{\delta}_{\nu c} = 0$ anyway, therefore the relative speeds of the engaged clutches will be kept bounded: $|\delta_{\nu c}| < \varepsilon$. If also the friction forces are not computed with sufficient precision it will be $\dot{\delta}_{\nu c} \simeq 0$ so a slow drift in the relative speeds appears. In this case the solution can be considered sufficiently accurate for most applications, moreover faster simulations are obtained if a precise zero crossing detection is not performed. This idea is closely related to the approach proposed in [Karnopp 1985]. Additional linear friction forces can be added to force $\delta_{\nu c}$ to zero.

5 Powershift gearbox example

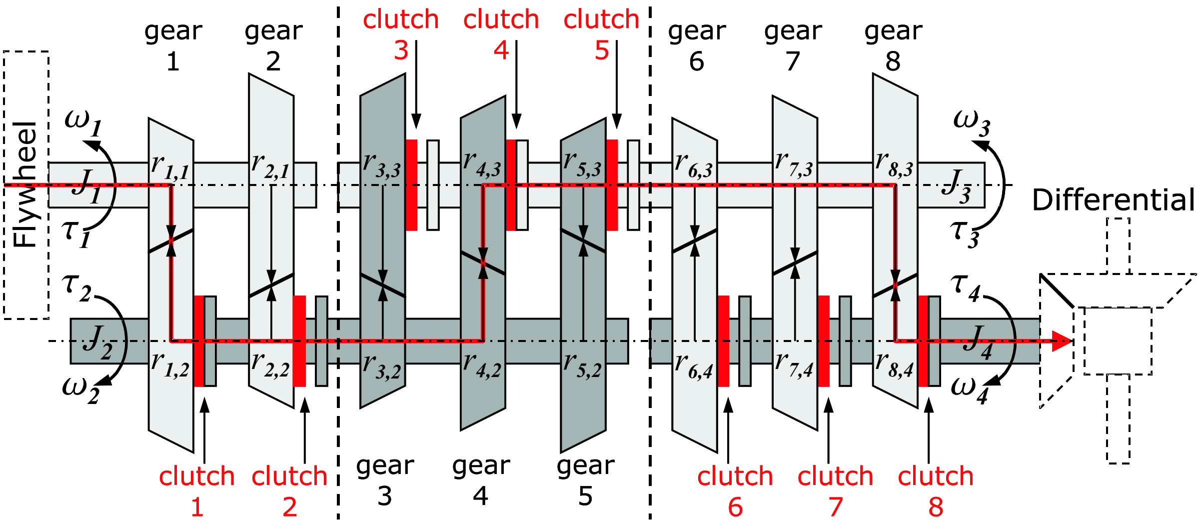

Figure 3: Mechanical scheme of the Powershift gearbox.

The Powershift gearbox is a feature of high level farm tractors. Its main feature is the capability to shift the gear ratio without torque interruption, an important advantage in agricultural activities. The framework of the Powershift gearbox is sketched in Fig. 3, it is similar to the automatic transmission of some cars, see [Jurghen 2000]. The engine flywheel is connected to the axle J_1, the two gear wheels with radius $r_{1,1}$ and $r_{2,1}$ rotate together with the axle J_1. The gear wheels $r_{1,1}$ and $r_{2,1}$ drive the two gear wheels $r_{1,2}$ and $r_{2,2}$ respectively. If the clutch 1 is engaged (as shown in Fig. 3) the axle J_2 is constrained to rotate at the same angular speed as the gear wheel $r_{1,2}$, by this way the two angular speeds ω_1 and ω_2 are constrained to rotate together with ratio $-r_{1,1}/r_{1,2}$:

$$r_{1,1}\,\omega_1 + r_{1,2}\,\omega_2 = 0 \qquad \Rightarrow \qquad \omega_2 = -\frac{r_{1,1}}{r_{1,2}}\,\omega_1$$

This concept can be repeated for all the gear wheel pairs. In Fig. 3 the clutches 1, 4, 8 are engaged, consequently the overall gear ratio from ω_1 to ω_4 is given by:

$$\left\{\begin{array}{rcl} r_{1,1}\,\omega_1 + r_{1,2}\,\omega_2 &=& 0 \\ r_{4,2}\,\omega_2 + r_{4,3}\,\omega_3 &=& 0 \\ r_{8,3}\,\omega_3 + r_{8,4}\,\omega_4 &=& 0 \end{array}\right. \qquad \Rightarrow \qquad \omega_4 = -\frac{r_{8,3}}{r_{8,4}}\,\frac{r_{4,2}}{r_{4,3}}\,\frac{r_{1,1}}{r_{1,2}}\,\omega_1 \qquad (20)$$

By changing the set of engaged clutches it is possible to change the static ratio between the velocities of the input shaft J_1 and the output shaft J_4. The number of possible gear ratios is $2 \cdot 3 \cdot 3 = 18$. The gear shift is executed disengaging some clutches and engaging some other new clutches. The simplest gear shift requires only a single swap (one clutch is engaged and one other is disengaged), the most demanding requires a triple swap. The control target is to smoothly shift the gear ratio while maintaining constant speed on the wheels or constant supplied torque by the engine.

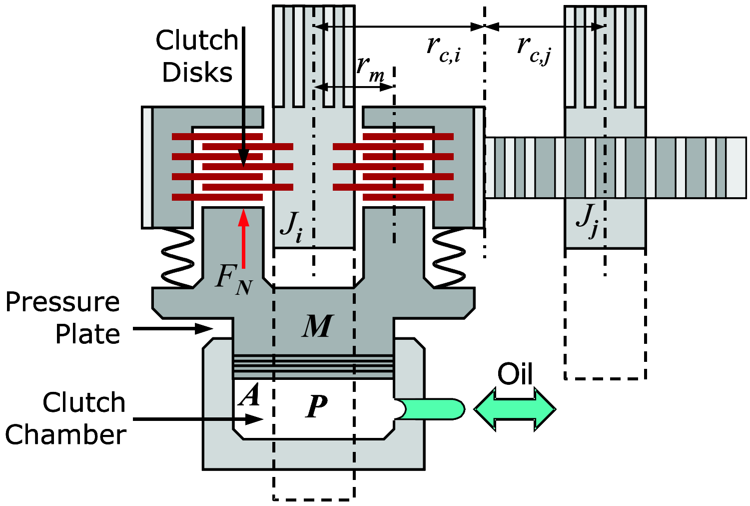

Figure 4: Simplified mechanical scheme of a wet clutch for a Powershift gearbox.

A simplified scheme of a clutch is shown in Fig. 4. By using an electrovalve it is possible to control the pressure P inside the clutch chamber and then the static and Coulomb friction amplitudes. For a detailed description please refer to [Morselli et al. 2003]. The issue is to find the proper pressure profiles inside the controlled clutches to obtain smooth gear shifts that satisfy the control target. The pressure P_i within the i^{th} clutch determines the force $F_{N,i}$ that presses the clutch disks. The friction amplitudes k_i and K_i to be used in the model (13) are given by:

$$K_i = F_{N,i}\,\eta_{s,i}\,n_{c,i}\,\frac{r_{m,i}}{r_{c,i}} \qquad\qquad k_i = F_{N,i}\,\eta_{c,i}\,n_{c,i}\,\frac{r_{m,i}}{r_{c,i}}$$

where $\eta_{s,i}$ is the static friction coefficient, $\eta_{c,i}$ is the Coulomb friction coefficient, $r_{m,i}$ is the clutch mean radius, $n_{c,i}$ is the number of friction surfaces and $r_{c,i}$ is the radius of the gear connected to the clutch. The simulations have been performed with $\eta_{s,i} = \eta_{c,i}$.

The simulation of the Powershift system is essential to develop and to verify by simulation the control strategies for the pressure profiles in order to obtain the control target. Moreover, a model of the system can be used to test the control software by *hardware in the loop* (HIL) experiments. To allow this kind of experiments, the model used for the simulations must be accurate but not computationally demanding.

The Powershift gearbox can also be seen as an hybrid system. It has 18 discrete states (all the possible gear ratios) and $n(n-1)/2 = 153$ possible transitions (it is possible to shift from one gear to any other one). The discrete states are quite simple, the tough problem is the management of all the possible transitions, this becomes even worse if the additional 6 reverse gears are taken into account. These arguments show that the hybrid approach is not suitable for the simulation of this system especially regarding HIL experiments.

Following the proposed approach, the Hamiltonian of the Powershift gearbox is:

$$H = \frac{1}{2}\,\omega^T\,\mathbf{M}\,\omega = \frac{1}{2}\,[\omega_1\ \omega_2\ \omega_3\ \omega_4]\begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & J_3 & 0 \\ 0 & 0 & 0 & J_4 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

The inertia J_i includes also the inertia of the wheel gears that rotate always together with the i-axle. The matrix $\mathbf{B}$ is a 4^{th} order identity matrix. The torques τ are given by:

$$\tau = \begin{bmatrix} \tau_M \\ 0 \\ 0 \\ \tau_L \end{bmatrix} + \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}$$

where τ_M is the engine torque, τ_L is the load torque and b_i are the viscous friction coefficients. The torques τ_M and τ_L and the coefficients b_i will be kept constant in the proposed simulations. For simplicity, the static and Coulomb friction amplitudes $\mathbf{K}$ and $\mathbf{k}$ are supposed to depend only on the normal force and the case $\mathbf{K} = \mathbf{k}$ ($\eta_{s,i} = \eta_{c,i}$) will be considered.

The matrix $\mathbf{R}$ takes the form:

$$\mathbf{R} = \begin{bmatrix} r_{1,1} & r_{1,2} & 0 & 0 \\ r_{2,1} & r_{2,2} & 0 & 0 \\ 0 & r_{3,2} & r_{3,3} & 0 \\ 0 & r_{4,2} & r_{4,3} & 0 \\ 0 & r_{5,2} & r_{5,3} & 0 \\ 0 & 0 & r_{6,3} & r_{6,4} \\ 0 & 0 & r_{7,3} & r_{7,4} \\ 0 & 0 & r_{8,3} & r_{8,4} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

For industrial reasons the numerical values are not referred to the real system.

Let's first analyze the meaning of the three cases described in Section 4 regarding the rank of the matrix $\mathbf{R}_\nu$ for the considered Powershif gearbox.

1) $\mathbf{R}_\nu$ is a full rank matrix with at most $n-1$ rows. Let $\nu = \{1, 4, 8\}$, the constraint forces λ can be computed form (16) and matrix $\mathbf{R}_\nu$ has exactly $3 = n-1$ rows. If K_ν is high enough, the clutches 1,4,8 are engaged, the system has only one degree of freedom and all the axles are constrained to rotate with constant speed ratios as computed in (20). This condition is shown in Fig. 3 and corresponds to a fixed gear ratio between the input axle and the output axle (normal operating condition).

2) $\mathbf{R}_\nu$ is a full rank $n \times n$ matrix. Let $\nu = \{1, 2, 4, 8\}$, the constraint $R_\nu\,\omega = 0$ allows only the trivial solution $\omega = 0$. If K_ν is high enough, the system does not have any degree of freedom an the axles are all stuck. Indeed, the contemporary engagement of the clutches 1 and 2 forces the axles J_1 and J_2 to stuck at zero speed, then the engaged clutches 3 and 4 constraint both the axles J_3 and J_4 at zero speed.

3) $\mathbf{R}_\nu$ is not a full rank matrix. Let $\nu = \{6, 7, 8\}$ and let K_ν high enough. The two axles J_3 and J_4 are stuck together, but there exist infinite solutions for λ in (16) and the system is over constrained, indeed only two engaged clutches between J_3 and J_4 are sufficient to ensure that J_3 and J_4 are stuck together. This condition is avoided by operating at most only two clutches at a time between each couple of axles.

Some simulation results regarding the Powershift gearbox are shown in Figs. 5÷10. The simulation is referred to an acceleration with constant engine torque τ_M and constant load torque τ_L. Fig. 5 shows the axles angular speeds and Fig. 6 shows the equivalent gear ratios between the pairs of axles. The gear radii expressed by matrix $\mathbf{R}$ ensure that when a clutch is engaged the corresponding gear ratio is an integer number. The overall gear ratio from ω_1 to ω_4 is shown in Fig. 7. Finally, the static and Coulomb friction forces are shown in Figs. 8÷10.

For $t = 0$ the clutches 1, 3 and 6 are engaged. The static friction forces keep the relative speeds δ_1, δ_3 and δ_6 at zero and they do not exceed the maximum static friction amplitude K_1, K_3 and K_6 as shown in Figs. 8÷10.

Around $t \sim 0.5\,\mathrm{s}$ a single swap gear shift is executed. The clutch 1 is disengaged while the clutch 2 is engaged. From $t = 0.5\,\mathrm{s}$ to $t = 0.7\,\mathrm{s}$ the clutches 1 and 2 are both slipping indeed the ratio ω_1/ω_2 is not integer and the friction forces equals the Coulomb friction amplitudes. The clutches 3 and 6 remain engaged even if the friction forces changes due to the gear shift. At the end of the gear shift the clutch 2 is engaged and its friction force (due to the static friction) is within the values $\pm K_2$.

For $t > 1.1\,\mathrm{s}$, the clutch 8 is slipping with $k_8 \neq 0$, this is just to test the model.

Around $t = 1.7\,\mathrm{s}$ a double swap gear shift is performed: the pairs of clutches 1-2 and 3-5 are swapped. Around $t = 2.5\,\mathrm{s}$ a triple swap gear shift is performed: the pairs of clutches 1-2, 4-5 and 6-7 are swapped. As shown in Fig. 7, the triple swap gear shift was not optimal: the overall gear ratio does not have a monotonic behaviour. The control of a triple swap gear shift is not a trivial task and the proposed model helps to find and to test the gearbox control strategies.

6 Conclusions

Static and Coulomb friction are extensively used in automotive mechanical systems to control the synchronization between two shafts or two axles. This paper have proposed a method for the efficient simulation of a wide class of automotive mechanical systems with static and Coulomb friction phenomena. The modeling approach is based on port-Hamiltonian systems, the computation of the friction forces or torques requires the zero crossing detection. If the hypothesis of an unique solution is possible, the computational effort becomes significantly lower. Without an accurate zero crossing detection a slight approximation is introduced, however the simulations are faster and sufficiently accurate for most applications. The proposed approach have been used to simulate the behavior of the Powershift gearbox provided with some high level farm tractors.

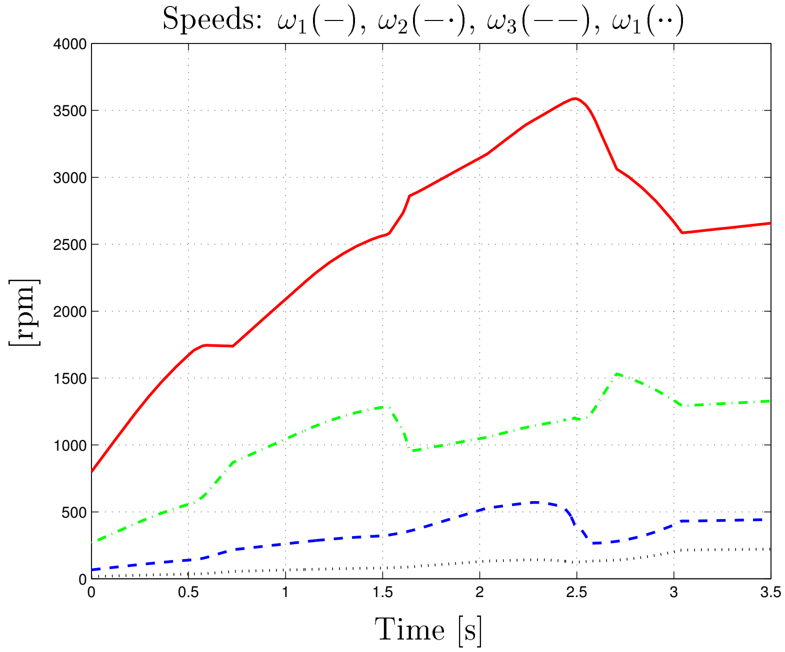

Figure 5: Axles angular speeds.

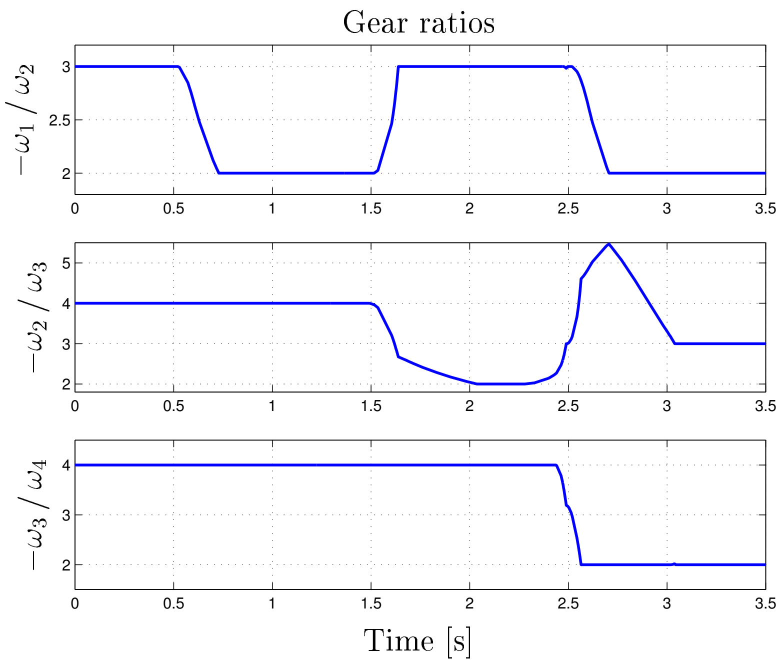

Figure 6: Speed ratios (reverse sign): from J_1 to J_2 (top), from J_2 to J_3 (center), from J_3 to J_4 (bottom).

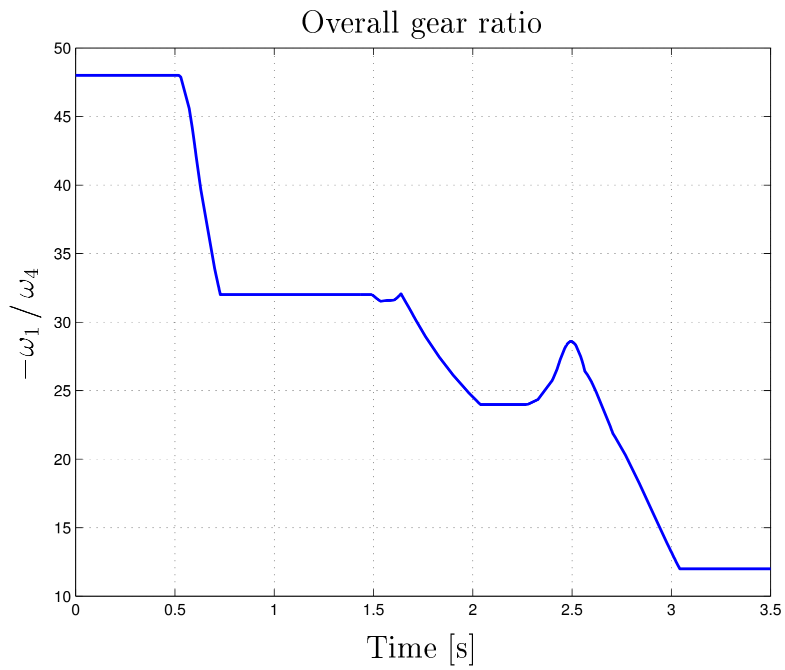

Figure 7: Equivalent overall gear ratio (reverse sign) from J_1 to J_4.

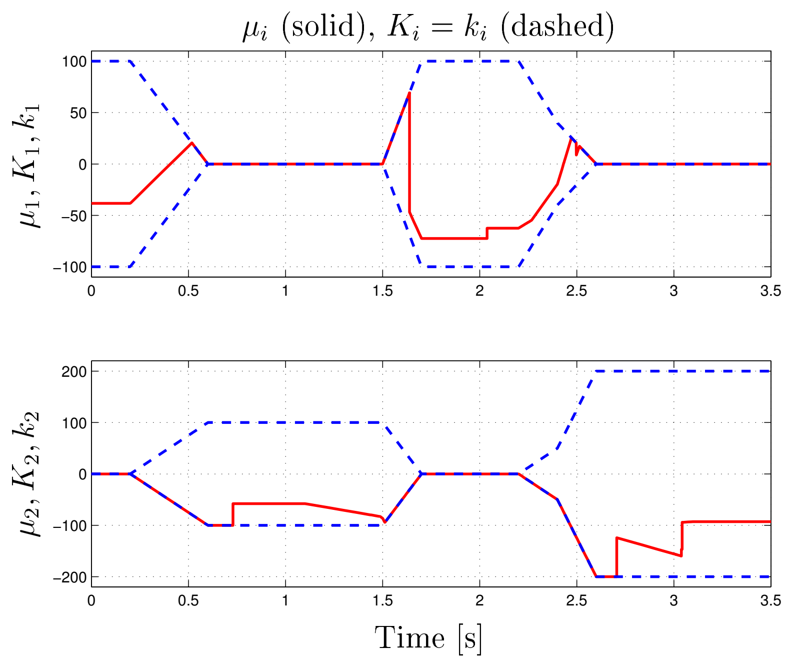

Figure 8: Static and Coulomb friction μ_i (solid) and Friction maximum amplitude $K_i = k_i$ (dashed) for the clutches 1 (top) and 2 (bottom).

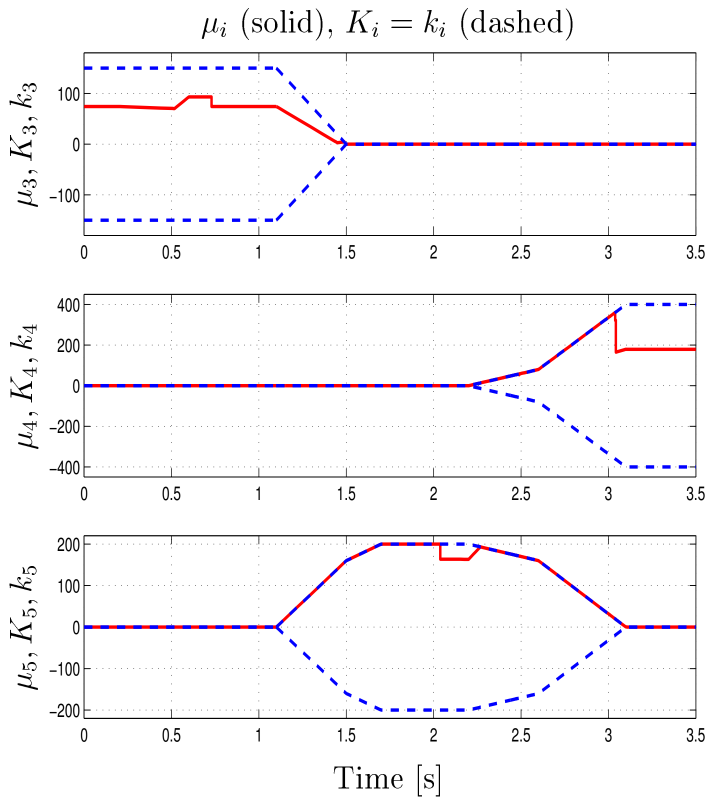

Figure 9: Static and Coulomb friction μ_i (solid) and Friction maximum amplitude $K_i = k_i$ (dashed) for the clutches 3 (top), 4 (center) and 5 (bottom).

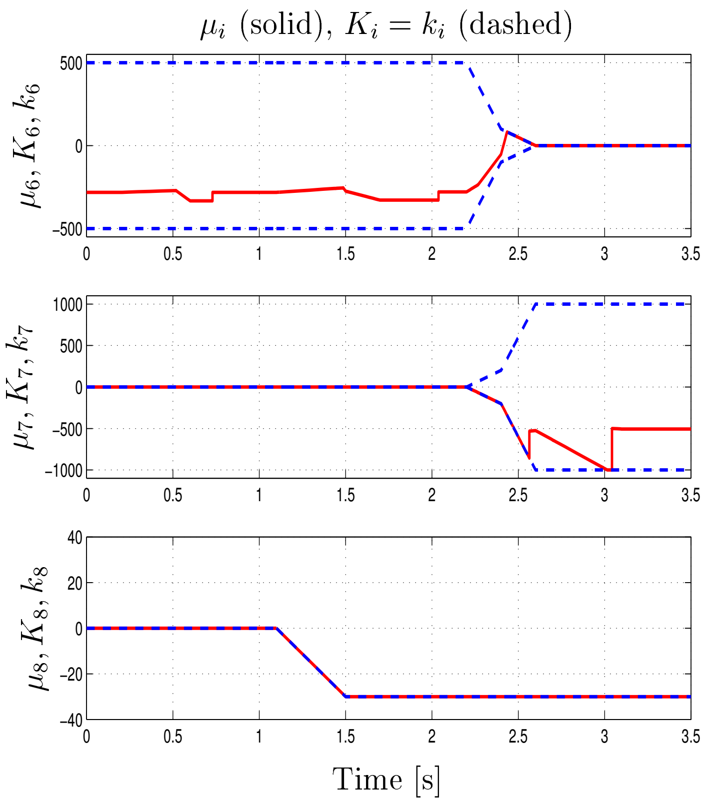

Figure 10: Static and Coulomb friction μ_i (solid) and Friction maximum amplitude $K_i = k_i$ (dashed) for the clutches 6 (top), 7 (center) and 8 (bottom).

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