Optimal Rotor Flux Shape for Multi-phase Permanent Magnet Synchronous Motors

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***Abstract*— In the paper the Power-Oriented Graphs (POG) technique is used for modeling m-phase permanent magnet synchronous motors and a study on the optimal rotor flux is given. The POG model shows the "power" internal structure of the considered electrical motor: the electric part interacts with the mechanical part by means of a "connection" block which neither stores nor dissipates energy. The dynamic model of the motor is as general as possible and it considers an arbitrary odd number of phases. The rotor flux is analyzed, in particular in order to minimize the currents needed for the torque generation, and its optimal shape is given. The model is finally implemented in Matlab/Simulink and the presented simulation results validate the machine model and the rotor flux choice.**

I. Introduction

The dynamic model of the multi-phase permanent magnet synchronous motors is known in literature obtained using classical mathematical methods. In this paper the dynamic model of these motors has been obtained using a Lagrangian approach in the frame of the Power-Oriented Graphs (POG) technique and, for the sake of generality, a generic periodic shape for the rotor flux has been considered. The obtained POG model is very compact, simple and puts in evidence the "power" internal structure of the motor. Using the POG approach and a Concordia-like transformation, the torque vector of the motor assumes a very simple structure which has been analyzed to find the optimal shape of the rotor flux minimizing the electrical power dissipation. The paper is organized as follows. Sec. II gives the basic properties of the POG modeling technique. Sec. III shows the details of POG dynamic model of the m-phase synchronous motors and the optimal shape of the rotor flux. Finally, in Sec. IV some simulation results are reported.

A. Notations

Row matrices will be denoted as follows:

$$\underset{1:n}{[\![}\; R_i \;\overset{i}{]\!]} = \begin{bmatrix} R_1 & R_2 & \dots & R_n \end{bmatrix},$$

column and diagonal matrices as:

$$\overset{i}{[\![}\; R_i \;\underset{1:n}{]\!]} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{bmatrix}, \quad \overset{i}{[\![}\; R_i \;\underset{1:n}{]\!]} = \begin{bmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_n \end{bmatrix}$$

and full matrices as:

$$\underset{1:n}{\overset{i}{[\![}}\; R_{i,j} \;\underset{1:m}{\overset{j}{]\!]}} = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nm} \end{bmatrix}$$

The symbol "$\sum_{n=a:d}^{b} c_n = c_a + c_{a+d} + c_{a+2d} + \dots$" will be used to represent the sum of a succession of numbers c_n where the index n ranges from a to b with increment d that is, using the Matlab symbology, $n = [a : d : b]$.

II. The bases of Power-Oriented Graphs

The POG technique [1] is a graphical modeling technique similar to Bond Graph (BG) [2], [3] and Energetic Macroscopic Representation (EMR) [4]. These techniques use the "power interaction" between subsystems as basic element for modeling. The two basic blocks used in the POG technique are shown in Fig. 1: the "elaboration block" (e.b.) and the "connection block" (c.b.). There

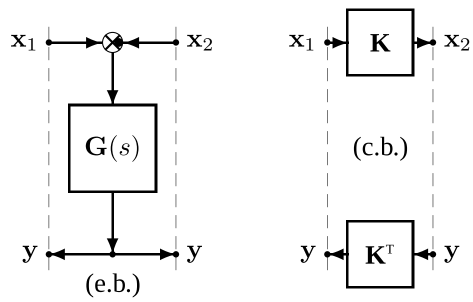

Fig. 1. The POG basic blocks: the elaboration block (e.b.) on the left and the connection block (c.b.) on the right.

is no restriction on the vector variables $\mathbf{x}$ and $\mathbf{y}$ other than the fact that their inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ must have the physical meaning of a "power". The e.b. is used for modeling all the physical elements that store and/or dissipate energy (springs, masses, dampers, etc.), i.e. all the 1-port elements (capacitors C, inertias I and resistor R) used in the BG technique. The c.b. is used for modeling all the physical elements that "transform the power" without losses (gear reductors, etc.), i.e. all the BG 2-port elements (transformers TR, gyrators GY, modulated transformers MTR and modulated gyrators MGY). The summation element at the top of the e.b. is used for modeling all the 3-port connection elements (0-junction and 1-junction) of the BG technique. More details on Power-Oriented Graphs are reported in [1], [5] and [6].

III. Electrical motors modelling

In this paper we will refer only to permanent magnet synchronous electrical motors with an *odd* number m of phases. The electromechanical structure of a seven-phase motor in the case of a single polar expansion ($p = 1$) is shown in Fig. 2. The considered multi-phase electrical motor is characterized by the following parameters:

m	: number of motor phases;
p	: number of polar expansions;
θ, θ_r	: electric and rotor angular positions: $\theta = p\,\theta_r$;
ω, ω_r	: electric and rotor angular velocities: $\omega = p\,\omega_r$;
N_c	: number of coils for each phase;
R_i	: i-th phase resistance ($p = 1$);
L_i	: i-th phase self induction coefficient ($p = 1$);
M_{ij}	: mutual induction coefficient of i-th phase coupled with j-th phase ($p = 1$);
$\phi(\theta)$	: rotor permanent magnet flux;
$\phi_c(\theta)$	: total rotor flux chained with stator phase 1;
$\phi_{ci}(\theta)$	: total rotor flux chained with stator phase i-th;
φ_r	: maximum value of function $\phi(\theta)$;
φ_c	: maximum value of function $\phi_c(\theta)$;
J_r	: rotor inertia momentum;
b_r	: rotor linear friction coefficient;
τ_r	: electromotive torque acting on the rotor;
τ_e	: external load torque acting on the rotor;
γ	: basic angular displacement;

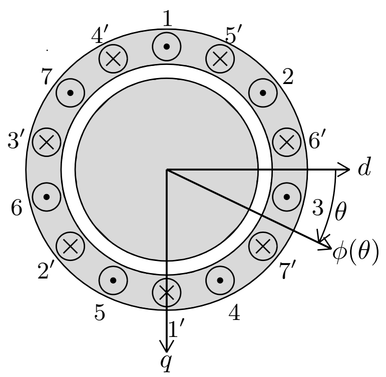

Fig. 2. Structure of a seven-phase motor in the case of single polar expansion ($p = 1$). Vectors d and q denote the "direct" and "quadrature" directions of the flux.

Fluxes $\phi(\theta)$ and $\phi_c(\theta)$ satisfy relations:

$$\phi_c(\theta) = p\,N_c\,\phi(\theta) = p\,N_c\,\varphi_r\,\bar{\phi}(\theta) = \varphi_c\,\bar{\phi}(\theta)$$

where $\varphi_c = p\,N_c\,\varphi_r$ and $\bar{\phi}(\theta)$ is the rotor flux function normalized with respect to its maximum value φ_r.

Let $\gamma = \frac{2\pi}{m}$ denote the basic angular phase displacement for electrical motors with m phases. The following hypotheses are assumed:

H1) Function $\phi(\theta)$ is periodic with period 2π;
H2) Function $\phi(\theta)$ is an even function of θ;
H3) Function $\phi(\theta + \frac{\pi}{2})$ is an odd function of θ;
H4) Flux $\phi_c(\theta)|_{\theta=0}$ chained with phase 1 is maximum;
H5) The motor electrical characteristics are homogeneous.

Let us introduce the state vectors $\dot{\mathbf{q}}$, $\mathbf{q}$ and the generalized flux vector $\mathbf{\Phi}(\mathbf{q})$:

$$\dot{\mathbf{q}} = \begin{bmatrix} \mathbf{I} \\ \omega \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} \mathbf{Q} \\ \theta \end{bmatrix}, \quad \mathbf{\Phi}(\mathbf{q}) = \begin{bmatrix} \mathbf{\Phi}_c(\theta) \\ 0 \end{bmatrix}$$

where $\mathbf{I}$ and $\mathbf{\Phi}_c$ are the current and flux vectors:

$$\mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_m \end{bmatrix}, \quad \mathbf{\Phi}_c(\theta) = \begin{bmatrix} \phi_{c1}(\theta) \\ \phi_{c2}(\theta) \\ \vdots \\ \phi_{cm}(\theta) \end{bmatrix} = \begin{bmatrix} \phi_c(\theta) \\ \phi_c(\theta-\gamma) \\ \vdots \\ \phi_c(\theta-(m-1)\gamma) \end{bmatrix}.$$

The dynamic equations of the electric motors can be obtained by using the following "Lagrangian" equation:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}^{\mathrm{T}}}\right) - \frac{\partial K}{\partial \mathbf{q}^{\mathrm{T}}} = \mathbf{V}_e - \mathbf{R}_e\,\dot{\mathbf{q}} \tag{1}$$

where K is the Lagrangian function of the system, $\mathbf{V}_e$ is the extended input vector and $\mathbf{R}_e$ is the extended dissipating matrix. For *multi-phase synchronous motors* the Lagrangian function K has the following structure:

$$K = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{L}_e\,\dot{\mathbf{q}} - \dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{\Phi}(\mathbf{q}) \tag{2}$$

where $\mathbf{L}_e$ is the extended energy matrix of the system. From (1) and (2) one obtains the dynamic equations:

$$\mathbf{L}_e\,\ddot{\mathbf{q}} = \mathbf{V}_e - \mathbf{R}_e\,\dot{\mathbf{q}} - \left[\frac{\partial \mathbf{\Phi}^{\mathrm{T}}}{\partial \mathbf{q}^{\mathrm{T}}} - \frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}}\right]\dot{\mathbf{q}}\,. \tag{3}$$

The term $\frac{\partial K}{\partial \mathbf{q}^{\mathrm{T}}}$ present in the left part of equation (1) represents the back electromotive voltage generated by the rotor movements. The last term of equation (3) is a skew-symmetric term which represents an internal energy redistribution. From (3) one directly obtains the differential equations of the motor:

$$\underbrace{\left[\begin{array}{c|c} \mathbf{L} & 0 \\ \hline 0 & J_r \end{array}\right]}_{\mathbf{L}_e} \underbrace{\left[\begin{array}{c} \dot{\mathbf{I}} \\ \hline \dot{\omega}_r \end{array}\right]}_{\ddot{\mathbf{q}}} = -\underbrace{\left[\begin{array}{c|c} \mathbf{R} & \mathbf{K}_\tau(\theta) \\ \hline -\mathbf{K}_\tau^{\mathrm{T}}(\theta) & b_r \end{array}\right]}_{\mathbf{R}_e + \mathbf{W}_e} \underbrace{\left[\begin{array}{c} \mathbf{I} \\ \hline \omega_r \end{array}\right]}_{\dot{\mathbf{q}}} + \underbrace{\left[\begin{array}{c} \mathbf{V} \\ \hline -\tau_e \end{array}\right]}_{\mathbf{V}_e} \tag{4}$$

Matrices $\mathbf{L} > 0$, $\mathbf{R}_e$ and $\mathbf{W}_e$ are defined as follows:

$$\mathbf{L} = p\left[\!\left[\; \overset{i}{\underset{1:m}{}}\; M_{ij} \;\overset{j}{\underset{1:m}{}} \;\right]\!\right], \quad \mathbf{R}_e = \left[\begin{array}{c|c} \mathbf{R} & 0 \\ \hline 0 & b_r \end{array}\right], \quad \mathbf{W}_e = \left[\begin{array}{c|c} 0 & \mathbf{K}_\tau(\theta) \\ \hline -\mathbf{K}_\tau^{\mathrm{T}}(\theta) & 0 \end{array}\right]$$

where $M_{ii} = L_i$, $M_{ij} = M_{ji}$ and terms $\mathbf{K}_\tau(\theta)$ and $\mathbf{R}$ are, respectively, the torque vector and the dissipating matrix:

$$\mathbf{K}_\tau(\theta) = \frac{\partial \mathbf{\Phi}_c^{\mathrm{T}}(\mathbf{q})}{\partial \theta}, \qquad \mathbf{R} = p\left[\!\left[\; \overset{i}{R_i} \;\right]\!\right]_{1:m}. \tag{5}$$

Function $\phi_c(\theta)$ is an even and periodic function of period 2π (see H1 and H2) and therefore it can be developed in Fourier series of cosines with only odd harmonics:

$$\phi_c(\theta) = \varphi_c\,\bar{\phi}(\theta) = \varphi_c \sum_{n=1:2}^{\infty} a_n\,\cos(n\theta). \tag{6}$$

Flux vector $\mathbf{\Phi}_c(\theta)$ can be rewritten in a compact form as:

$$\mathbf{\Phi}_c(\theta) = \varphi_c \left|\left|\left[\!\left[\; \sum_{n=1:2}^{\infty} a_n \cos[n(\theta - h\,\gamma)] \;\right]\!\right]\right|\right|_{0:m-1}^{h}. \tag{7}$$

From (5), the torque vector $\mathbf{K}_\tau(\theta)$ can be expressed as follows:

$$\mathbf{K}_\tau(\theta) = p\,\varphi_c \left\| \left[-\sum_{n=1:2}^{\infty} n\,a_n \sin[n(\theta - h\,\gamma)] \right] \right\|_{0:m-1}^{h}. \qquad (8)$$

Let us now consider the following orthonormal transformation:

$${}^t\mathbf{T}_\omega^{\mathrm{T}} = {}^\omega\mathbf{T}_t = \sqrt{\frac{2}{m}} \begin{bmatrix} \left\| \left\| \begin{bmatrix} \cos(k\,(\theta - h\,\gamma)) \\ \sin(k\,(\theta - h\,\gamma)) \end{bmatrix} \right\|_{1:2:m-2}^{k} \right\|_{0:m-1}^{h} \\ \left\| \left[\frac{1}{\sqrt{2}} \right] \right\|_{0:m-1}^{h} \end{bmatrix}.$$

Matrix ${}^\omega\mathbf{T}_t$ represents a multi-dimensional rotation in the state space. Matrix ${}^\omega\mathbf{T}_t(\theta)$ is a function of the electrical angle θ and transforms the electric variables $\mathbf{V}$ and $\mathbf{I}$ from the original reference frame Σ_t to a transformed rotating frame Σ_ω. Applying transformation ${}^\omega\mathbf{T}_t$ to matrices $\mathbf{L}$, $\mathbf{R}$ and $\mathbf{K}_\tau(\theta)$, from (4) one obtains the following transformed system:

$$\left[\begin{array}{c|c} {}^\omega\mathbf{L} & 0 \\ \hline 0 & J_r \end{array}\right] \left[\begin{array}{c} {}^\omega\dot{\mathbf{I}} \\ \hline \dot{\omega}_r \end{array}\right] = -\left[\begin{array}{c|c} {}^\omega\mathbf{R} + \mathbf{J}_\omega\,{}^\omega\mathbf{L} & {}^\omega\mathbf{K}_\tau \\ \hline -\,{}^\omega\mathbf{K}_\tau^{\mathrm{T}} & b_r \end{array}\right] \left[\begin{array}{c} {}^\omega\mathbf{I} \\ \hline \omega_r \end{array}\right] + \left[\begin{array}{c} {}^\omega\mathbf{V} \\ \hline -\tau_e \end{array}\right] \qquad (9)$$

where ${}^\omega\mathbf{I} = {}^\omega\mathbf{T}_t\,\mathbf{I}$, ${}^\omega\mathbf{V} = {}^\omega\mathbf{T}_t\,\mathbf{V}$, ${}^\omega\mathbf{R} = {}^\omega\mathbf{T}_t\,\mathbf{R}\,{}^t\mathbf{T}_\omega = \mathbf{R} = p\,R\,\mathbf{I}_m$ and ${}^\omega\mathbf{L} = {}^\omega\mathbf{T}_t\,\mathbf{L}\,{}^t\mathbf{T}_\omega$. Let the self and mutual induction coefficients of matrix $\mathbf{L}$ be defined as:

$$\begin{cases} M_{i\,j} = M_0 \cos((i-j)\gamma) \\ L_i = \Delta_0 + M_0 \end{cases} \quad \text{for} \;\; i, j \in \{1, 2, ..., m\}\,.$$

The transformed matrix ${}^\omega\mathbf{L}$ has the following structure:

$${}^\omega\mathbf{L} = p \begin{bmatrix} \Delta_0 + \frac{m\,M_0}{2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \Delta_0 + \frac{m\,M_0}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Delta_0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \Delta_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \Delta_0 \end{bmatrix},$$

where Δ_0 and M_0 are proper positive parameters. The structure of transformed vector ${}^\omega\mathbf{K}_\tau(\theta)$ is the following:

$${}^\omega\mathbf{K}_\tau(\theta) = {}^\omega\mathbf{T}_t\,\mathbf{K}_\tau(\theta) = -p\,\varphi_c \sqrt{\frac{m}{2}} \cdot$$
$$\cdot \begin{bmatrix} \left\| \begin{bmatrix} \sum_{n=0:2m}^{\infty} [(n+k)\,a_{n+k} + (n-k)\,a_{n-k}] \sin(n\theta) \\ \sum_{n=0:2m}^{\infty} [(n+k)\,a_{n+k} - (n-k)\,a_{n-k}] \cos(n\theta) \end{bmatrix} \right\|_{1:2:m-2}^{k} \\ -\sqrt{2} \sum_{n=m:2m}^{\infty} n a_n \sin(n\theta) \end{bmatrix} \qquad (10)$$

Vector ${}^\omega\mathbf{K}_\tau(\theta)$ can be easily computed knowing the coefficients a_n of the Fourier series of the rotor flux, see eq. (6). Note that vector ${}^\omega\mathbf{K}_\tau(\theta)$ is composed only by the harmonics $\sin(n\theta)$ and $\cos(n\theta)$ where n is an integer number multiple of $2m$. A detailed discussion of the properties of the components of vector ${}^\omega\mathbf{K}_\tau(\theta)$ can be found in [8]. The structure of matrix $\mathbf{J}_\omega$ and vector ${}^\omega\mathbf{I}$ in (9) are the following:

$$\mathbf{J}_\omega = \begin{bmatrix} \left\| \begin{bmatrix} 0 & -k\,\omega \\ k\,\omega & 0 \end{bmatrix} \right\|_{1:2:m-2}^{k} & 0 \\ 0 & 0 \end{bmatrix}, \quad {}^\omega\mathbf{I} = \begin{bmatrix} \left\| \begin{bmatrix} {}^\omega I_{dk} \\ {}^\omega I_{qk} \end{bmatrix} \right\|_{1:2:m-2}^{k} \\ {}^\omega I_m \end{bmatrix}$$

where $\omega = \dot{\theta}$ is the time-derivative of the electric angle θ and ${}^\omega I_{dk}$, ${}^\omega I_{qk}$ are, respectively, the *direct* and *quadrature* components of the current vector ${}^\omega\mathbf{I}$.

The POG scheme of the multi-phase electrical motor in the transformed space Σ_ω, see eq. (9), is shown in Fig. 3. The elaboration blocks present between the power sections ① and ② represent the *Electrical part* of the system, while the blocks present between sections ③ and ④ represent the *Mechanical part* of the system. The connection blocks present between sections ① and ⓘ and between sections ② and ③ represent, respectively, the state space transformation ${}^t\mathbf{T}_\omega$ between reference frames Σ_t and Σ_ω, and the energy and power conversion (without accumulation nor dissipation) between the electrical and mechanical parts of the motor.

Proposition 1: the torque vector ${}^\omega\mathbf{K}_\tau(\theta)$ in (10) is constant (i.e. it is not function of the electric angle θ) only for the flux functions $\bar{\phi}(\theta)$ which can be expressed in Fourier series as follows:

$$\bar{\phi}(\theta) = \sum_{i=1:2}^{m-2} a_i \cos(i\,\theta) \qquad (11)$$

Proof. From (10) it follows that the torque vector ${}^\omega\mathbf{K}_\tau(\theta)$ is constant and different from zero if and only if the following relations hold:

$$\begin{cases} a_{kq}(n,\,k) \neq 0 & \text{for}\; n=0 \text{ and } k \in \{1:2:m-2\} \\ \begin{pmatrix} a_{kd}(n,\,k) = 0 \\ a_{kq}(n,\,k) = 0 \end{pmatrix} & \text{for} \begin{pmatrix} n \in \{2m : 2m : \infty\} \\ k \in \{1:2:m-2\} \end{pmatrix} \\ a_n = 0 & \text{for}\; n \in \{m : 2m : \infty\} \end{cases} \qquad (12)$$

where $a_{kq}(n,\,k) = [(n+k)\,a_{n+k} - (n-k)\,a_{n-k}]$, $a_{kd}(n,\,k) = [(n+k)\,a_{n+k} + (n-k)\,a_{n-k}]$. Since $n > k$ when $n \neq 0$ and $a_h = 0$ when $h < 0$, relations (12) can be rewritten in the following equivalent form:

$$\begin{cases} a_k \neq 0 & \text{for} \quad k \in \{1:2:m-2\} \\ \begin{pmatrix} a_{n+k} = 0 \\ a_{n-k} = 0 \end{pmatrix} & \text{for} \begin{pmatrix} n \in \{2m : 2m : \infty\} \\ k \in \{1:2:m-2\} \end{pmatrix} \\ a_n = 0 & \text{for} \quad n \in \{m : 2m : \infty\} \end{cases} \qquad (13)$$

that is

$$a_i \neq 0 \qquad \text{for} \qquad i \in \{1:2:m-2\}$$

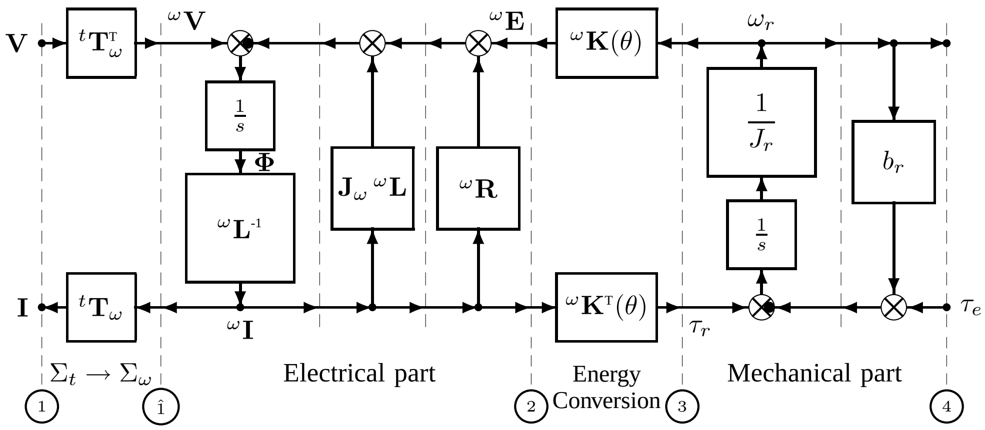

Fig. 3. POG scheme of a multi-phase electrical motor in the transformed space Σ_ω.

as stated in (11), so *Proposition 1* is proved. □

Remark 1: flux function $\bar{\phi}(\theta)$ in (11) is valid only when the torque vector ${}^\omega\mathbf{K}_\tau(\theta)$ has the form given in (10) which corresponds to the case of m independent motor phases. In the opposite case of star-connected phases, the last term of vector ${}^\omega\mathbf{K}_\tau(\theta)$ is not present and relation (11) must be substituted by $\tilde{\phi}(\theta) = \bar{\phi}(\theta) + \bar{\phi}_m(\theta)$ where the new term $\bar{\phi}_m(\theta) = \sum_{i=m:2m}^{\infty} a_i \cos(i\,\theta)$ has no influence on the torque generation.

All the possible constant vectors ${}^\omega\mathbf{K}_\tau(\theta)$ are obtained from (10) when $n = 0$:

$$ {}^\omega\mathbf{K}_\tau^{\mathrm{T}} = {}^\omega\mathbf{K}_\tau^{\mathrm{T}}(\theta)|_{n=0} = -\varphi_c\, p\sqrt{\frac{m}{2}}\left[\|[\ 0 \quad \overset{k}{k\, a_k}\ ]\|_{1:2:m-2}\ 0 \right]. \quad (14) $$

<u>*Main Proposition:*</u> *among all the fluxes that provide a constant vector* ${}^\omega\mathbf{K}_\tau$*, see (11), the one that minimizes the current module (and therefore the dissipated power) is given by:*

$$ \bar{\phi}(\theta) = a_{m-2}\cos((m-2)\,\theta). \quad (15) $$

Proof. Let ${}^\omega\mathbf{I}_d$ denote the constant desired current. The condition ${}^\omega\mathbf{I} = {}^\omega\mathbf{I}_d$ can be achieved by using the following control law:

$$ {}^\omega\mathbf{V} = ({}^\omega\mathbf{R} + \mathbf{J}_\omega\,{}^\omega\mathbf{L})\,{}^\omega\mathbf{I} + {}^\omega\mathbf{K}_\tau\,\omega_r - \mathbf{K}_s({}^\omega\mathbf{I} - {}^\omega\mathbf{I}_d) \quad (16) $$

where $\mathbf{K}_s > 0$ is a diagonal matrix used for the tuning of the control. Putting relation (16) in (9) one obtains the dynamics:

$$ {}^\omega\mathbf{L}\,{}^\omega\dot{\mathbf{I}} = -\mathbf{K}_s({}^\omega\mathbf{I} - {}^\omega\mathbf{I}_d)\,. \quad (17) $$

Defining ${}^\omega\tilde{\mathbf{I}} = ({}^\omega\mathbf{I} - {}^\omega\mathbf{I}_d)$ as the current error vector and remembering that ${}^\omega\mathbf{L}\,{}^\omega\dot{\mathbf{I}}_d = 0$, from (17) one obtains

$$ {}^\omega\mathbf{L}\,{}^\omega\dot{\tilde{\mathbf{I}}} = -\mathbf{K}_s\,{}^\omega\tilde{\mathbf{I}}\,. $$

With a proper choice of matrix $\mathbf{K}_s$ it is possible to give the subsystems the desired dynamics. The time constant τ_i of the i-th subsystem is given by

$$ \tau_i = \frac{{}^\omega L_i}{K_{s_i}} $$

where ${}^\omega L_i$ and K_{s_i} are the i-th element on the diagonal of matrix ${}^\omega\mathbf{L}$ and matrix $\mathbf{K}_s$, respectively. Chosen the desired time constants, matrix $\mathbf{K}_s$ is given by

$$ \mathbf{K}_s = \overset{i}{[\![\ K_{s_i}\ ]\!]}_{1:m}\,, \qquad K_{s_i} = \frac{{}^\omega L_i}{\tau_i}\,. \quad (18) $$

Since matrix $\mathbf{K}_s$ is positive definite, after a transient the error vector ${}^\omega\tilde{\mathbf{I}}$ tends asymptotically to zero (i.e. ${}^\omega\mathbf{I}$ tends to ${}^\omega\mathbf{I}_d$) for all the desired constant currents ${}^\omega\mathbf{I}_d$. The desired torque τ_d generated by current vector ${}^\omega\mathbf{I}_d$ is given by relation:

$$ \tau_d = {}^\omega\mathbf{K}_\tau^{\mathrm{T}}\,{}^\omega\mathbf{I}_d\,. \quad (19) $$

The set of all the current vectors ${}^\omega\mathbf{I}_d$ satisfying relation (19) is the following:

$$ {}^\omega\mathbf{I}_d = {}^\omega\mathbf{I}_0 + \mathrm{Ker}[{}^\omega\mathbf{K}_\tau^{\mathrm{T}}] \quad (20) $$

where ${}^\omega\mathbf{I}_0$ is a particular solution of system (19) and $\mathrm{Ker}[{}^\omega\mathbf{K}_\tau^{\mathrm{T}}]$ is the kernel of the row matrix ${}^\omega\mathbf{K}_\tau^{\mathrm{T}}$. Among all the vectors ${}^\omega\mathbf{I}_d$ given by (20) the one which has the minimum modulus is the current vector ${}^\omega\mathbf{I}_d$ which is parallel to vector ${}^\omega\mathbf{K}_\tau$:

$$ {}^\omega\mathbf{I}_d = \frac{\tau_d}{|{}^\omega\mathbf{K}_\tau|}\,{}^\omega\hat{\mathbf{K}}_\tau \quad (21) $$

where ${}^\omega\hat{\mathbf{K}}_\tau$ denotes the versor of vector ${}^\omega\mathbf{K}_\tau$. Note that the modulus of current ${}^\omega\mathbf{I}_d$ is inversely proportional to the modulus of vector ${}^\omega\mathbf{K}_\tau$. So, among all the fluxes given by (11), the one that minimizes the modulus of the current ${}^\omega\mathbf{I}_d$ is the one which maximizes the modulus $|{}^\omega\mathbf{K}_\tau|$ of vector ${}^\omega\mathbf{K}_\tau$. From (14) it follows that the problem of finding the $(m-1)/2$ coefficients a_k that maximize the modulus of vector ${}^\omega\mathbf{K}_\tau$ is equivalent to the problem of maximizing the following functional $F(\mathbf{a})$:

$$ F(\mathbf{a}) = \sqrt{\sum_{k=1:2}^{m-2}(k\,a_k)^2} $$

where $\mathbf{a} = \{a_1,\, a_3,\, \ldots,\, a_{m-2}\}$, under the constraint of unitary maximum amplitude of the flux function $\bar{\phi}(\theta, \mathbf{a})$:

$$ \max_\theta \bar{\phi}(\theta, \mathbf{a}) = 1. \quad (22) $$

This problem admits one single maximum for:

$$a_k = \begin{cases} 0 & \text{for} \quad k = [1:2:m-4] \\ 1 & \text{for} \quad k = m-2 \end{cases}$$

which corresponds to the flux shape $\bar{\phi}(\theta) = a_{m-2}\cos((m-2)\,\theta)$ given in (15). So, for constant torque τ_r the rotor flux $\bar{\phi}(\theta)$ which minimizes the modulus of the motor current ${}^{\omega}\mathbf{I}_d$ (least power dissipation) is the flux (15). □

In the case of $m = 7$, the functional $F(\mathbf{a})$ is a function of parameters a_1, a_3 and a_5. Since these parameters satisfy constraint (22), then it is always possible to express two of them as a function of the third one. Fig. 4 shows the functional $F(\mathbf{a})$ as a function of the normalized parameters a_1/a_5 and a_3/a_5: the maximum corresponds to solution $a_1 = a_3 = 0$ and $a_5 = 1$.

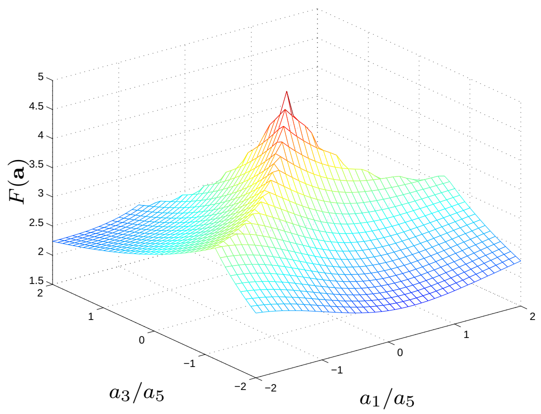

Fig. 4. Functional $F(\mathbf{a})$ as a function of parameters a_k in the case of $m = 7$.

Remark 2: eq. (14) clearly shows that all the odd components of the torque vector ${}^{\omega}\mathbf{K}_\tau$ are zero and from (21) it follows that the minimum vector ${}^{\omega}\mathbf{I}_d$ that generates the desired torque τ_d is parallel to the torque vector ${}^{\omega}\mathbf{K}_\tau$. This leads to conclude that the *direct* components ${}^{\omega}I_{dk}$ of vector ${}^{\omega}\mathbf{I}_d$ must be zero.

IV. Simulations

The Simulink scheme of the controlled electric motor is shown in Fig. 5: the main central block corresponds to the POG scheme shown in Fig. 3. The simulation results presented in this Section have been obtained using the following electrical and mechanical parameters: $m = 9$, $p = 1$, $R = 3\,\Omega$, $L_0 = 0.1$ H, $M_0 = 0.08$ H, $N_c = 30$, $\varphi_r = 0.02$ W, $J_r = 0.5$ kg m^2, $b_r = 1.8$ N m s/rad and $\tau_e = 0$ Nm. The odd harmonics $\{1, 3, 5, 7\}$ of the rotor flux $\bar{\phi}(\theta)$ when $m = 9$ and $p = 1$ are shown in Fig. 6. The motor phases are supposed to be star connected. The input vector ${}^{\omega}\mathbf{V}$ is given by control law (16) where ${}^{\omega}\mathbf{I}_d$ has been calculated using relation (19) considering the desired torque $\tau_d = 10$ Nm for $t \in [0, 1.5]$ s and $\tau_d = 5$ Nm for $t \in [1.5, 3]$ s. The elements K_{s_i} of matrix $\mathbf{K}_s$, see eq. (18), are chosen in order to have the time constants $\tau_i = \{0.33\text{s}, 0.25\text{s}, 0.17\text{s}, 0.09\text{s}\}$ for $i \in \{1, 3, 5, 7\}$. Figures 7÷10 show simulation results

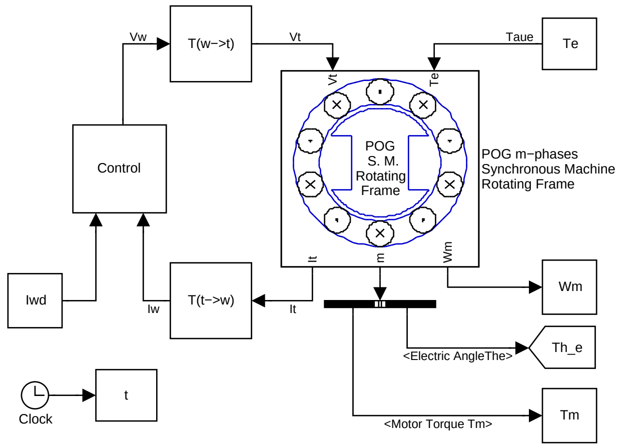

Fig. 5. Simulink scheme of the controlled electric motor.

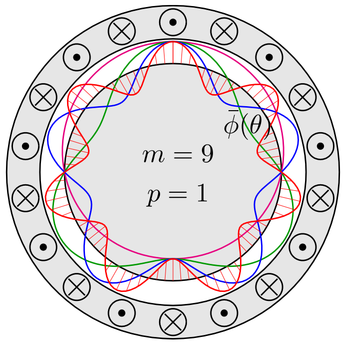

Fig. 6. Shapes of the rotor flux $\bar{\phi}(\theta)$: odd harmonics $\{1, 3, 5, 7\}$ when $m = 9$ and $p = 1$.

obtained for different harmonics of the rotor flux: 1-st harmonic (magenta), 3-rd harmonic (green), 5-th harmonic (blue) and 7-th harmonic (red). Fig. 7 shows the motor velocity ω_r and the rotor torque τ_r: note that the desired value τ_d is reached with the time constants τ_i defined above. Fig. 8 shows the modulus of the phase currents ${}^{\omega}\mathbf{I}$: note that the smallest current corresponds to the 7-th harmonic (red line) of the rotor flux, as stated in (15). The quadrature currents ${}^{\omega}I_{qk}$, for $k \in \{1, 3, 5, 7\}$, are shown in Fig. 9. Direct currents are not shown because they are equal to zero. Figures 10 and 11 are obtained with the same parameters and same control, but with saturated input voltages $|V_i| \le 14$ V. Fig. 10 shows the motor velocity ω_r and the rotor torque τ_r: for growing values of the torque vector, the counter electromotive torques increase and the reached velocities decrease. Saturated voltages $\mathbf{V}$ and currents $\mathbf{I}$ in the original reference frame Σ_t for the 7-th harmonic are shown in Fig. 11: note that when the input voltages V_i reach the saturation the desired torque τ_d is no more obtained and a small ripple appears on the final value, see the zoom in Fig. 10.

V. Conclusions

In this paper a m-phase permanent magnet synchronous motor has been modeled using the Power-Oriented Graphs (POG) technique. The obtained POG model is very compact and can be easily implemented in Simulink.

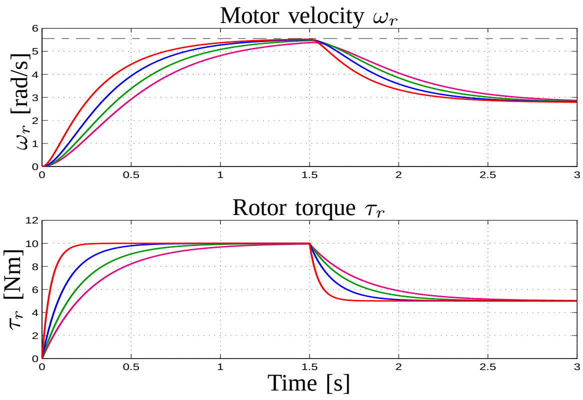

Fig. 7. Motor velocity ω_r and rotor torque τ_r for the odd harmonics $\{1, 3, 5, 7\}$ of the rotor flux $\bar{\phi}(\theta)$.

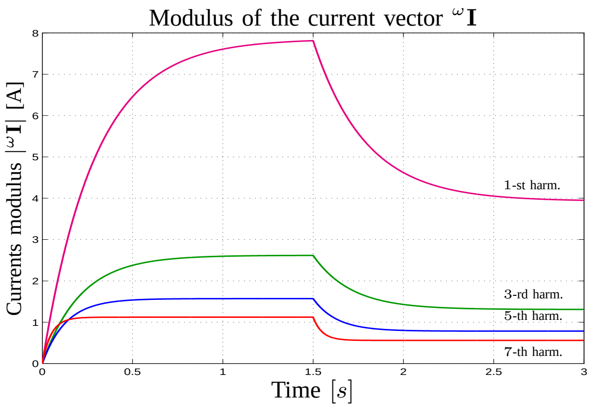

Fig. 8. Modulus of the phase currents ${}^{\omega}\mathbf{I}$ in the transformed reference frame Σ_ω.

A deep analysis of the torque vector has been carried out in order to find the optimal shape of the rotor flux which minimizes the module of the current vector. Simulation results show the effectiveness of the realized model in the case of a nine-phase star-connected motor.

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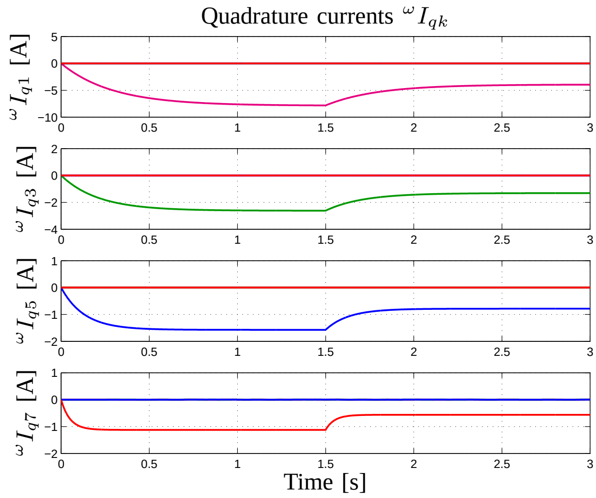

Fig. 9. Quadrature currents ${}^{\omega}I_{qk}$ in the transformed reference frame Σ_ω for the harmonics $i = \{1, 3, 5, 7\}$ of the rotor flux $\bar{\phi}(\theta)$. Currents ${}^{\omega}I_{qk}$ are different from zero only when $k = i$.

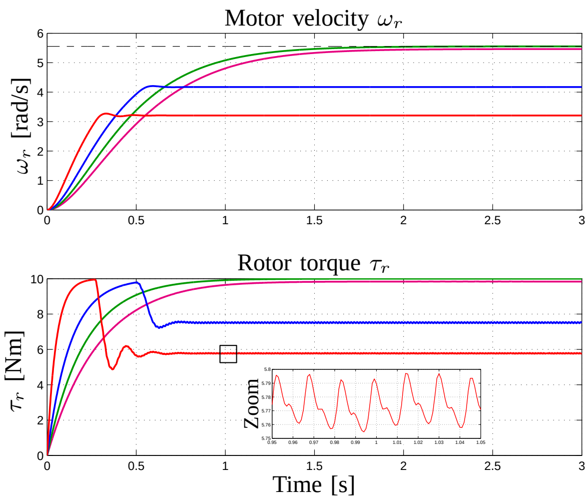

Fig. 10. Motor velocity ω_r and rotor torque τ_r for the odd harmonics $\{1, 3, 5, 7\}$ of the rotor flux $\bar{\phi}(\theta)$ when the input voltage is saturated.

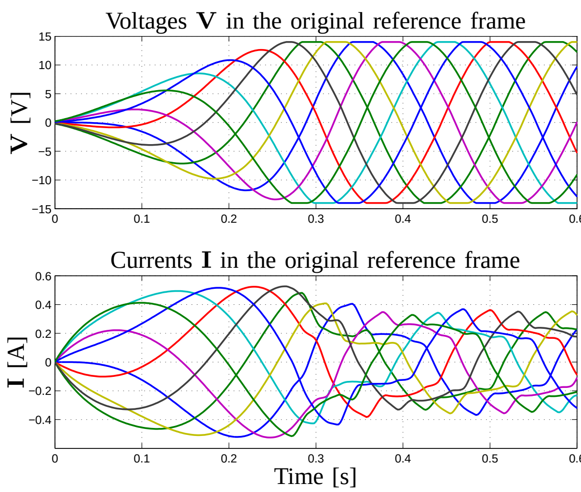

Fig. 11. Voltages $\mathbf{V}$ and currents $\mathbf{I}$ in the original reference frame Σ_t for the 7-th harmonic when the input voltages are saturated to 14 V.